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1997 J. Phys. A: Math. Gen. 30 6029

(http://iopscience.iop.org/0305-4470/30/17/014)

J. Phys. A: Math. Gen. **30** (1997) 6029–6055. Printed in the UK PII: S0305-4470(97)79719-1

Lichnerowicz–Jacobi cohomology

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Received 20 November 1996

Abstract. In this paper we extend the notion of Lichnerowicz–Poisson cohomology to Jacobi manifolds. We study the relation of the so-called Lichnerowicz–Jacobi cohomology with the basic de Rham cohomology and the cohomology of the Lie algebra of functions relative to the representation defined by the Hamiltonian vector fields. A natural pairing with the canonical homology is constructed. The relation between the Lichnerowicz–Poisson cohomology of a quantizable Poisson manifold and the Lichnerowicz–Jacobi cohomology of the total space of a prequantization bundle is obtained. Particular cases of cosymplectic, contact and locally conformal symplectic manifolds are discussed. Finally, the Lichnerowicz–Jacobi cohomology of a non-transitive example is studied.

1. Introduction

Poisson and Jacobi manifolds have gained much attention since their introduction by Lichnerowicz ([21, 22], see also [2, 20, 31]). The most appealing fact concerning these manifolds is the existence of a bracket of functions satisfying the Jacobi's identity. In the case of Poisson manifolds the history goes back to Lagrange, Poisson, Hamilton and Lie (see [31, 32]). Indeed, a Poisson bracket is introduced in mechanics in order to put in a canonical way the motion equations. Lichnerowicz showed that a Poisson bracket $\{\,,\,\}$ on the algebra of functions on a manifold M actually comes from a skew-symmetric tensor field Λ of type $(2, 0)$ in such a way that the Jacobi identity is equivalent to the vanishing of the Schouten–Nijenhuis bracket $[\Lambda, \Lambda]$. The relation between both formulations is provided by the formula $\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g)$, which is a link between the algebra of functions on M and the geometry of the manifold.

More general types of brackets were discussed by Shiga for the space of the sections of a vector bundle, and for Kirillov for line bundles [23, 14]. These brackets have a local nature (the property to be derivations in each argument is lost) and they lead to the notion of Jacobi bracket. Again, it was Lichnerowicz [22] who interpreted this notion in a geometrical way. A Jacobi structure on a manifold M consists of a pair (Λ, E), where Λ is a skew-symmetric $(2, 0)$-type tensor field and E is a vector field on M such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$ and $[E, \Lambda] = 0$. The manifold M endowed with a Jacobi structure is called a Jacobi manifold. The relation with the Jacobi bracket of functions is provided by the formula $\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f)$.

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It is obvious that a Jacobi structure with $E = 0$ is Poisson. Thus, Poisson (and hence symplectic) manifolds are the first examples of Jacobi manifolds. But there are many examples of strictly non-Poisson Jacobi manifolds. For instance, contact and locally conformal symplectic manifolds are Jacobi. In fact, Lichnerowicz has proved that any Jacobi manifold possesses a generalized foliation whose leaves are contact or locally conformal symplectic manifolds. In particular, a Poisson manifold is foliated by symplectic leaves.

In [21] Lichnerowicz studied the Chevalley–Eilenberg cohomology of the Lie algebra of the functions on a Poisson manifold (M, Λ), with special emphasis on the so-called 1-differentiable cohomology. Let us recall that a k-cochain in the Chevalley–Eilenberg complex is called 1-differentiable if it is defined by a first-order differential operator on functions. An interesting subcomplex of the 1-differentiable complex is that which consists of pure 1-differentiable cochains, that is, those cochains defined from skew-symmetric contravariant tensor fields. The resultant cohomology operator on the contravariant Grassmann algebra on M is given by $\sigma(P) = -[\Lambda, P]$, and the associated cohomology is called the Lichnerowicz–Poisson (LP) cohomology. This cohomology is related to the de Rham cohomology, but it is of a different nature. In fact, if M is symplectic they are isomorphic, but the result does not hold for arbitrary Poisson manifolds. Even more, the computation of the LP-cohomology may be very difficult (see [31]). The operator σ plays a key role in the geometric quantization procedure developed by Vaisman [30, 31] for Poisson manifolds, which extends the well known Konstant–Souriau one. Indeed, since we have to handle contravariant tensors instead of differential forms, we need to use the operator σ instead of the exterior differential.

On the other hand, in [22] Lichnerowicz studied the Chevalley–Eilenberg cohomology on the Lie algebra of the functions on a Jacobi manifold (M, Λ, E). He related the cohomology of the subcomplex of the 1-differentiable cochains with the LP-cohomology of the poissonization of M (the tangentially exact Poisson manifold associated with M in the terminology of Lichnerowicz) and he proved that in some cases the tangent 1-differentiable cohomology is trivial. He also obtained the derivations of the different algebras associated to M.

However, there is an alternative cohomology associated with M: the cohomology of the Lie algebra of the functions relative to the representation defined by the Hamiltonian vector fields (see section 4). We call it H–Chevalley–Eilenberg cohomology and it was used by Vaisman [30] to study the prequantization representations of the Lie algebra of functions on a Jacobi manifold. This fact leads us to believe that this cohomology may be used to construct the quantization of a Jacobi manifold. It should be noted that the H–Chevalley–Eilenberg cohomology of a Poisson manifold coincides with its usual Chevalley–Eilenberg cohomology.

The purpose of this paper is to extend and study the cohomology operator σ to Jacobi manifolds. If (M, Λ, E) is a Jacobi manifold we define $\sigma(P) = -[\Lambda, P] + kP$, for a k-vector P. The restriction of σ to the skew-symmetric contravariant tensor fields which are invariant with respect to E is the desired extension. The resultant cohomology is called Lichnerowicz–Jacobi (LJ) cohomology of M. If M is Poisson it coincides with the LP-cohomology. The LJ-cohomology of M is isomorphic to the cohomology of a subcomplex of the H–Chevalley–Eilenberg complex, and it is also related to the basic de Rham cohomology of M (with respect to E).

The paper is organized as follows. After some introductory definitions and results in section 2, we define the LJ-cohomology of a Jacobi manifold in section 3. Its relation with the H–Chevalley–Eilenberg cohomology is ellucidated in section 4. In section 5 we relate the LJ-cohomology with the basic de Rham cohomology with respect to E. In section 6 we define a natural pairing between the LJ-cohomology and the canonical homology introduced in [5, 7]. This pairing extends the one for Poisson manifolds studied in [2]. The relation between the LP-cohomology of a quantizable Poisson manifold and the LJ-cohomology of the Jacobi structure induced on the total space of a prequantization bundle is discussed in section 7. In sections 8–10, we study in detail the LJ-cohomology of a cosymplectic manifold and of the transitive Jacobi manifolds: contact and locally conformal symplectic manifolds. We end the paper by studying the LJ-cohomology of a non-transitive example.

2. Jacobi and Poisson manifolds

All the manifolds considered in this paper are assumed to be connected.

A *Jacobi structure* on a m-dimensional manifold M is a pair (Λ, E) where Λ is a 2-vector and E a vector field on M satisfying the following properties:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda \qquad [E, \Lambda] = 0. \tag{1}$$

Here [,] denotes the Schouten–Nijenhuis bracket. The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. A bracket of functions (the *Jacobi bracket*) is defined by

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + fE(g) - gE(f) \qquad \text{for all } f, g \in C^\infty(M, \mathbb{R}). \tag{2}$$

The Jacobi bracket { , } is skew symmetric, satisfies the Jacobi identity and

$$\text{support}\{f, g\} \subset (\text{support } f) \cap (\text{support } g).$$

Thus, the space $C^\infty(M, \mathbb{R})$ of C^∞ real-valued functions on M endowed with the Jacobi bracket is *a local Lie algebra* in the sense of Kirillov [14]. Conversely, a structure of local Lie algebra on $C^\infty(M, \mathbb{R})$ defines a Jacobi structure on M (see [12, 14]). If the vector field E vanishes then { , } is a derivation in each argument and, therefore { , } defines a *Poisson bracket* on M. In this case, (1) reduces to

$$[\Lambda, \Lambda] = 0 \tag{3}$$

and (M, Λ) is a *Poisson manifold*.

Examples of Poisson manifolds are symplectic and cosymplectic manifolds.

A *symplectic manifold* is a pair $(\bar{M}, \bar{\Omega})$, where $\bar{M}$ is an even-dimensional manifold and $\bar{\Omega}$ is a closed non-degenerate 2-form on $\bar{M}$. We define a Poisson 2-vector $\bar{\Lambda}$ on $\bar{M}$ by

$$\bar{\Lambda}(\bar{\alpha}, \bar{\beta}) = \bar{\Omega}(\bar{\flat}^{-1}(\bar{\alpha}), \bar{\flat}^{-1}(\bar{\beta})) \tag{4}$$

for all $\bar{\alpha}, \bar{\beta} \in \Omega^1(\bar{M})$, where $\Omega^1(\bar{M})$ is the space of 1-forms on $\bar{M}$ and $\bar{\flat} : \mathfrak{X}(\bar{M}) \longrightarrow \Omega^1(\bar{M})$ is the isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules defined by $\bar{\flat}(\bar{X}) = i_{\bar{X}}\bar{\Omega}$.

A *cosymplectic manifold* (see [19]) is a triple $(\bar{M}, \bar{\Phi}, \bar{\eta})$, where $\bar{M}$ is an odd-dimensional manifold, $\bar{\Phi}$ is a closed 2-form and $\bar{\eta}$ is a closed 1-form on $\bar{M}$ such that $\bar{\eta} \wedge \bar{\Phi}^m$ is a volume form, with $\dim \bar{M} = 2m + 1$. If $\bar{\flat} : \mathfrak{X}(\bar{M}) \longrightarrow \Omega^1(\bar{M})$ is the isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules defined by $\bar{\flat}(\bar{X}) = i_{\bar{X}}\bar{\Phi} + (i_{\bar{X}}\bar{\eta})\bar{\eta}$, then the vector field $\bar{\xi} = \bar{\flat}^{-1}(\bar{\eta})$ is called the *Reeb vector field* of $\bar{M}$. The vector field $\bar{\xi}$ is characterized by the relations $i_{\bar{\xi}}\bar{\Phi} = 0$ and $i_{\bar{\xi}}\bar{\eta} = 1$. A 2-vector $\bar{\Lambda}$ on $\bar{M}$ is defined by

$$\bar{\Lambda}(\bar{\alpha}, \bar{\beta}) = \bar{\Phi}(\bar{\flat}^{-1}(\bar{\alpha}), \bar{\flat}^{-1}(\bar{\beta})) = \bar{\Phi}(\bar{\flat}^{-1}(\bar{\alpha} - \bar{\alpha}(\bar{\xi})\bar{\eta}), \bar{\flat}^{-1}(\bar{\beta} - \bar{\beta}(\bar{\xi})\bar{\eta})) \tag{5}$$

for all $\bar{\alpha}, \bar{\beta} \in \Omega^1(\bar{M})$. Thus, $(\bar{M}, \bar{\Lambda})$ becomes a Poisson manifold.

Other interesting examples of Jacobi manifolds, which are not Poisson manifolds, are the contact manifolds and the locally conformal symplectic manifolds which we will describe below.

Let M be a $(2m+1)$-dimensional manifold and θ be a 1-form on M. We said that θ is a contact 1-form if $\theta \wedge (\mathrm{d}\theta)^m \neq 0$ at every point. In such a case (M, θ) is called a *contact manifold* (see, for example, [3]). A contact manifold (M, θ) is a Jacobi manifold. In fact, we define the 2-vector Λ on M by

$$\Lambda(\alpha, \beta) = \mathrm{d}\theta(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \tag{6}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X \,\mathrm{d}\theta + \theta(X)\theta$. The vector field E is just the Reeb vector field $\xi = \flat^{-1}(\theta)$ of (M, θ). We remark that $i_\xi \theta = 1$ and $i_\xi \,\mathrm{d}\theta = 0$.

An *almost symplectic manifold* is a pair (M, Ω), where M is an even-dimensional manifold and Ω is a non-degenerate 2-form on M. An almost symplectic manifold is said to be *locally conformal symplectic (l.c.s.)* if for each point $x \in M$ there is an open neighbourhood U such that $\mathrm{d}(\mathrm{e}^{-\sigma}\Omega) = 0$, for some function $\sigma : U \longrightarrow \mathbb{R}$. If $U = M$ then M is said to be a *globally conformal symplectic (g.c.s.)* manifold (see for example [29]). An almost symplectic manifold (M, Ω) is l.(g.)c.s. if and only if a closed (exact) 1-form ω exists such that

$$\mathrm{d}\Omega = \omega \wedge \Omega. \tag{7}$$

The 1-form ω is called the *Lee 1-form* of M. It is obvious that the l.c.s. manifolds with Lee 1-form identically zero are just the symplectic manifolds. We define a 2-vector Λ and a vector field E by

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \qquad E = \flat^{-1}(\omega) \tag{8}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$. Then (M, Λ, E) is a Jacobi manifold. Note that

$$\omega(E) = 0 \qquad \mathcal{L}_E\omega = 0 \qquad \mathcal{L}_E\Omega = 0. \tag{9}$$

The contact and l.c.s. manifolds are called the *transitive Jacobi manifolds* (see [8]).

Now, let (M, Λ, E) be a Jacobi manifold. Define a mapping $\# : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ from the space of 1-forms $\Omega^1(M)$ on M onto the Lie algebra $\mathfrak{X}(M)$ of the vector fields on M as follows

$$(\#\alpha)(\beta) = \Lambda(\alpha, \beta) \tag{10}$$

for $\alpha, \beta \in \Omega^1(M)$.

Remark 2.1. For a contact manifold M with Reeb vector field ξ, we have that $\#(\alpha) = -\flat^{-1}(\alpha) + \alpha(\xi)\xi$. For a l.c.s. manifold M, we obtain that $\# = -\flat^{-1}$. Finally, for a cosymplectic manifold $\bar{M}$ with Reeb vector field $\bar{\xi}$, we deduce that $\bar{\#}(\bar{\alpha}) = -\bar{\flat}^{-1}(\bar{\alpha}) + \bar{\alpha}(\bar{\xi})\bar{\xi}$.

If f is a C^∞ real-valued function on a Jacobi manifold M, the vector field X_f defined by

$$X_f = \#(\mathrm{d}f) + fE \tag{11}$$

is called the *Hamiltonian vector field* associated with f. It should be noted that the Hamiltonian vector field associated with the constant function 1 is just E. A direct computation proves that $[X_f, X_g] = X_{\{f,g\}}$ (see [22, 24]).

Now, for every $x \in M$, we consider the subspace D_x of T_xM generated by all the Hamiltonian vector fields evaluated at the point x. In other words, $D_x = \#_x(T_x^*M) + \langle E_x \rangle$. Since D is involutive, one easily follows that D defines a generalized foliation, which is called the *characteristic foliation* in [8]. Moreover, the Jacobi structure of M induces a transitive Jacobi structure on each leaf L (for a more detailed study of the characteristic foliation of a Jacobi manifold we refer to [8]). If M is a Poisson manifold then, from (10) and (11), we deduce that the characteristic foliation of M is just the *canonical symplectic foliation* of M (see [21, 32]).

A Jacobi manifold (M, Λ, E) is said to be *regular* if the vector field E is complete, $E \neq 0$ at every point and the one-dimensional foliation defined by E is regular in the sense of Palais [25]. In such a case, the space of leaves $\bar{M} = M/E$ has a structure of a differentiable manifold and the canonical projection $\pi : M \longrightarrow \bar{M}$ is a fibration (surjective submersion). Moreover, we can define a 2-vector $\bar{\Lambda}$ on $\bar{M}$ by

$$\bar{\Lambda}(\bar{\alpha}, \bar{\beta}) \circ \pi = \Lambda(\pi^*\bar{\alpha}, \pi^*\bar{\beta}) \qquad \forall \bar{\alpha}, \bar{\beta} \in \Omega^1(\bar{M}).$$

Note that, from (1), $\bar{\Lambda}$ is well defined and $(\bar{M}, \bar{\Lambda})$ is a Poisson manifold (see [8]).

3. Lichnerowicz–Jacobi cohomology

Let $(\bar{M}, \bar{\Lambda})$ be a Poisson manifold and $\mathcal{V}^k(\bar{M})$ the space of k-vectors on $\bar{M}$. The differential operator $\bar{\sigma} : \mathcal{V}^k(\bar{M}) \longrightarrow \mathcal{V}^{k+1}(\bar{M})$ given by

$$\bar{\sigma}(\bar{P}) = -[\bar{\Lambda}, \bar{P}] \tag{12}$$

defines a cohomology operator on $\mathcal{V}^*(\bar{M}) = \oplus_k \mathcal{V}^k(\bar{M})$. The cohomology of the corresponding differential complex $(\mathcal{V}^*(\bar{M}), \bar{\sigma})$ is denoted by $H^*_{\mathrm{LP}}(\bar{M})$ and called the *LP-cohomology* of $\bar{M}$ (for more details, see [21]). Note that $\bar{\sigma}(\bar{\Lambda}) = 0$ and thus $\bar{\Lambda}$ defines a cohomology class in $H^2_{\mathrm{LP}}(\bar{M})$. This fact is important, for instance, in the geometric quantization of Poisson manifolds (see [30, 31]; see also [13]).

Now, let (M, Λ, E) be a Jacobi manifold. We define the differential operator $\sigma : \mathcal{V}^k(M) \longrightarrow \mathcal{V}^{k+1}(M)$ as follows (see [18])

$$\sigma(P) = -[\Lambda, P] + kE \wedge P \qquad \text{for } P \in \mathcal{V}^k(M). \tag{13}$$

From a straightforward computation, using (1) and (13), we conclude the following.

Proposition 3.1. If (M, Λ, E) is a Jacobi manifold, σ is the differential operator given by (13) and $\mathcal{L}_E$ denotes the Lie derivative with respect to E, then:

$$\mathcal{L}_E \circ \sigma = \sigma \circ \mathcal{L}_E \tag{14}$$

$$\sigma^2(P) = -\mathcal{L}_E P \wedge \Lambda \qquad \text{for } P \in \mathcal{V}^k(M) \tag{15}$$

$$\sigma(P \wedge Q) = \sigma(P) \wedge Q + (-1)^k P \wedge \sigma(Q) \qquad \text{for } P \in \mathcal{V}^k(M) \text{ and } Q \in \mathcal{V}^*(M). \tag{16}$$

Denote by $\mathcal{V}^k_I(M)$ the subspace of $\mathcal{V}^k(M)$ defined by

$$\mathcal{V}^k_I(M) = \{P \in \mathcal{V}^k(M) / \mathcal{L}_E P = [E, P] = 0\}$$

that is, $\mathcal{V}^k_I(M)$ is the subspace of invariant k-vectors with respect to the vector field E. It is clear that $\mathcal{V}^0_I(M)$ is the space $C^\infty_B(M, \mathbb{R}) = \{f \in C^\infty(M, \mathbb{R}) / E(f) = 0\}$ of basic functions on M. Moreover, $\mathcal{V}^k_I(M)$ is a $C^\infty_B(M, \mathbb{R})$-module. Also, from proposition 3.1, the following corollary follows.

Corollary 3.2. Let (M, Λ, E) be a Jacobi manifold and σ the differential operator given by (13). If $P \in \mathcal{V}^k_I(M)$, then $\sigma(P) \in \mathcal{V}^{k+1}_I(M)$ and $\sigma^2(P) = 0$.

This last result allows us to introduce the differential complex

$$\cdots \longrightarrow \mathcal{V}^{k-1}_I(M) \xrightarrow{\sigma_I} \mathcal{V}^k_I(M) \xrightarrow{\sigma_I} \mathcal{V}^{k+1}_I(M) \longrightarrow \cdots$$

where $\sigma_I = \sigma_{|\mathcal{V}_I^*(M)}$ and $\mathcal{V}_I^*(M) = \oplus_k \mathcal{V}_I^k(M)$. This complex defines a cohomology which is called *the LJ-cohomology* of (M, Λ, E). The kth LJ-cohomology group is then given by

$$H_{\mathrm{LJ}}^k(M) = \frac{\ker\{\sigma_I : \mathcal{V}_I^k(M) \longrightarrow \mathcal{V}_I^{k+1}(M)\}}{\mathrm{Im}\{\sigma_I : \mathcal{V}_I^{k-1}(M) \longrightarrow \mathcal{V}_I^k(M)\}}.$$

Note that $\Lambda \in \mathcal{V}_I^2(M)$, $E \in \mathcal{V}_I^1(M)$, $\sigma_I(\Lambda) = 0$ and $\sigma_I(E) = 0$. Consequently, Λ and E define cohomology classes in $H_{\mathrm{LJ}}^2(M)$ and $H_{\mathrm{LJ}}^1(M)$, respectively. On the other hand, using (16) we have that $\wedge$ induces an associative product in $H_{\mathrm{LJ}}^*(M) = \oplus_k H_{\mathrm{LJ}}^k(M)$. Also, it is clear that if M is a Poisson manifold then the LJ-cohomology of M is just the LP-cohomology.

Below we will describe an interesting subcomplex of $(\mathcal{V}_I^*(M), \sigma_I)$.

Let $\mathcal{V}_{IE}^k(M)$ be the subspace of $\mathcal{V}_I^k(M)$ defined by

$$\mathcal{V}_{IE}^k(M) = \{P \in \mathcal{V}^k(M)/\mathcal{L}_E P = 0 \text{ and } E \wedge P = 0\}.$$

Since $\sigma_I(E) = 0$, then σ_I induces a subcomplex $(\mathcal{V}_{IE}^*(M), \sigma_{IE})$, where $\sigma_{IE} = (\sigma_I)_{|\mathcal{V}_{IE}^*(M)}$ and $\mathcal{V}_{IE}^*(M) = \oplus_k \mathcal{V}_{IE}^k(M)$. The cohomology of this subcomplex is denoted by $H_{\mathrm{LJ}E}^*(M)$ and called the *E-LJ-cohomology* of M.

The following result relates the E-LJ-cohomology and the LJ-cohomology of M.

Proposition 3.3. Let (M, Λ, E) be a Jacobi manifold and θ a 1-form on M such that $\theta(E) = 1$ and $\mathcal{L}_E\theta = 0$. Then:

(i) there is an exact sequence of complexes

$$0 \longrightarrow (\mathcal{V}_{IE}^*(M), \sigma_{IE}) \stackrel{i}{\longrightarrow} (\mathcal{V}_I^*(M), \sigma_I) \stackrel{\pi}{\longrightarrow} (\mathcal{V}_{IE}^{*+1}(M), -\sigma_{IE}) \longrightarrow 0$$

where $i : \mathcal{V}_{IE}^*(M) \longrightarrow \mathcal{V}_I^*(M)$ is the inclusion map and $\pi : \mathcal{V}_I^*(M) \longrightarrow \mathcal{V}_{IE}^{*+1}(M)$ is the epimorphism defined by $\pi(Q) = E \wedge Q$, for every $Q \in \mathcal{V}_I^*(M)$.

(ii) This exact sequence induces a long exact cohomology sequence

$$\cdots \longrightarrow H_{\mathrm{LJ}E}^k(M) \stackrel{i_k^*}{\longrightarrow} H_{\mathrm{LJ}}^k(M) \stackrel{\pi_k^*}{\longrightarrow} H_{\mathrm{LJ}E}^{k+1}(M) \stackrel{\Delta_k^*}{\longrightarrow} H_{\mathrm{LJ}E}^{k+1}(M) \longrightarrow \cdots$$

with connecting homomorphism Δ^*.

Futher results on the E-LJ-cohomology of M will be obtained in section 7.

4. Lichnerowicz–Jacobi cohomology and H–Chevalley–Eilenberg cohomology

First, we recall the definition of the cohomology of a Lie algebra $\mathcal{A}$ with values (or coefficients) in an $\mathcal{A}$-module $\mathcal{M}$ (we will follow [31]).

Let $(\mathcal{A}, [\ ,\ ])$ be a real Lie algebra (not necessarily finite dimensional) and $\mathcal{M}$ be a linear space endowed with a bilinear multiplication $\mathcal{A} \times \mathcal{M} \longrightarrow \mathcal{M}$ such that $[a_1, a_2]m = a_1(a_2m) - a_2(a_1m)$. In such a case, one says that $\mathcal{M}$ is an *$\mathcal{A}$-module relative to the given representation of $\mathcal{A}$ on $\mathcal{M}$*. Then, a k-linear skew-symmetric mapping $c^k : \mathcal{A}^k \longrightarrow \mathcal{M}$ is called an *$\mathcal{M}$-valued k-cochain* and these cochains form a linear space $C^k(\mathcal{A}; \mathcal{M})$. The formula

$$\begin{aligned}(\partial_k c^k)(a_0, \ldots, a_k) &= \sum_{i=0}^{k} (-1)^i a_i c^k(a_0, \ldots, \widehat{a_i}, \ldots, a_k) \\ &\quad + \sum_{i<j} (-1)^{i+j} c^k([a_i, a_j], a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_k)\end{aligned}$$

defines a coboundary since, as for the exterior differential d, $\partial_{k+1} \circ \partial_k = 0$. Hence, we have the corresponding cohomology spaces

$$H^k(\mathcal{A}; \mathcal{M}) = \frac{\ker\{\partial_k : C^k(\mathcal{A}; \mathcal{M}) \to C^{k+1}(\mathcal{A}; \mathcal{M})\}}{\mathrm{Im}\{\partial_{k-1} : C^{k-1}(\mathcal{A}; \mathcal{M}) \to C^k(\mathcal{A}; \mathcal{M})\}}.$$

This cohomology is called *the cohomology of the Lie algebra $\mathcal{A}$ with values (or coefficients) in $\mathcal{M}$, or relative to the given representation of $\mathcal{A}$ on $\mathcal{M}$*. Because of the Jacobi identity, any Lie algebra $(\mathcal{A}, [\,,\,])$ is an $\mathcal{A}$-module, to be denoted by ($\mathcal{A}$-ad), for the operation $a.a' = [a, a']$. The cochains of $\mathcal{A}$ with values in $\mathcal{A}$-ad are called *Chevalley–Eilenberg cochains*, and the corresponding cohomology spaces are the *Chevalley–Eilenberg cohomology spaces of $\mathcal{A}$*.

Next, we will study the relation between the LJ-cohomology of a Jacobi manifold M and the cohomology of the Lie algebra $(C^\infty(M, \mathbb{R}), \{,\})$ relative to the representation of $C^\infty(M, \mathbb{R})$ on $C^\infty(M, \mathbb{R})$ defined by the Hamiltonian vector fields.

Let (M, Λ, E) be a Jacobi manifold and $(C^\infty(M, \mathbb{R}), \{\,,\,\})$ the Lie algebra of the C^∞ real-valued functions on M endowed with the Jacobi bracket. Using the fact that $[X_f, X_g] = X_{\{f,g\}}$ for $f, g \in C^\infty(M, \mathbb{R})$, we deduce that $C^\infty(M, \mathbb{R})$ is a $C^\infty(M, \mathbb{R})$-module relative to the representation

$$C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}) \qquad (f, g) \longrightarrow X_f(g).$$

The cohomology of the Lie algebra $(C^\infty(M, \mathbb{R}), \{\,,\,\})$ relative to the above representation defined by the Hamiltonian vector fields is called the *H–Chevalley–Eilenberg cohomology of M*. We will denote by $C^k_{HCE}(M)$ to the k-cochains in the *H–Chevalley–Eilenberg complex* of M, by ∂_H to the *H–Chevalley–Eilenberg cohomology operator* and by $H^k_{HCE}(M)$ to the kth H–Chevalley–Eilenberg cohomology group.

Remark 4.1. If M is a Poisson manifold then, since $X_f(g) = \{f, g\}$ for $f, g \in C^\infty(M, \mathbb{R})$, we deduce that the H–Chevalley–Eilenberg cohomology of M is just the Chevalley–Eilenberg cohomology.

Now, let $i : \mathcal{V}^k(M) \longrightarrow C^k_{HCE}(M)$ be the monomorphism of real vector spaces given by

$$i(P)(f_1, \ldots, f_k) = P(\mathrm{d}f_1, \ldots, \mathrm{d}f_k) \tag{17}$$

for all $P \in \mathcal{V}^k(M)$ and $f_1, \ldots, f_k \in C^\infty(M, \mathbb{R})$.

A direct computation, using (2), (10), (11) and (13), proves the following.

Proposition 4.2. Let (M, Λ, E) be a Jacobi manifold and ∂_H be the H–Chevalley–Eilenberg cohomology operator. Suppose that $P \in \mathcal{V}^k(M)$, that $i : \mathcal{V}^k(M) \longrightarrow C^k_{HCE}(M)$ is the monomorphism defined by (17) and that $\sigma : \mathcal{V}^k(M) \longrightarrow \mathcal{V}^{k+1}(M)$ is the differential operator given by (13).

(i) If $f_0, \ldots, f_k \in C^\infty(M, \mathbb{R})$ then,

$$(\partial_H(\mathrm{i}(P)) - \mathrm{i}(\sigma(P)))(f_0, \ldots, f_k) = \sum_{j=0}^{k} (-1)^j f_j(\mathrm{i}(\mathcal{L}_E P))(f_0, \ldots, \widehat{f_j}, \ldots, f_k).$$

(ii) $\mathcal{L}_E P = 0$ if and only if $\partial_H(\mathrm{i}(P)) = \mathrm{i}(\sigma(P))$.

The above result suggests introducing the following definition.

Definition 4.3. Let (M, Λ, E) be a Jacobi manifold and $\tilde{P} \in C^k_{HCE}(M)$. The k-cochain $\tilde{P}$ is said to be pure 1-differentiable if there exists $P \in \mathcal{V}^k_I(M)$ such that $\tilde{P} = \mathrm{i}(P)$.

Remark 4.4.

(i) Definition 4.3 generalizes for Jacobi manifolds the notion of pure 1-differentiable k-cochain in a Poisson manifold (see [21]).

(ii) If (M, Λ, E) is a Jacobi manifold and $\tilde{P} \in C^k_{HCE}(M)$ then $\tilde{P}$ is pure 1-differentiable if and only if for $f_1, g_1, f_2, \ldots, f_k \in C^\infty(M, \mathbb{R})$ we have

$$\tilde{P}(f_1 g_1, f_2, \ldots, f_k) = f_1 \tilde{P}(g_1, f_2, \ldots, f_k) + g_1 \tilde{P}(f_1, f_2, \ldots, f_k)$$
$$E(\tilde{P}(f_1, \ldots, f_k)) - \sum_{j=1}^{k} \tilde{P}(f_1, \ldots, E(f_j), \ldots, f_k) = 0.$$

Denote by $\tilde{C}^k_{HCE}(M)$ the subspace of pure 1-differentiable k-cochains on a Jacobi manifold (M, Λ, E). Proposition 4.2 and the results of section 3 allow us to introduce the following subcomplex of the H–Chevalley–Eilenberg complex of M:

$$\cdots \longrightarrow \tilde{C}^{k-1}_{HCE}(M) \xrightarrow{\tilde{\partial}_H} \tilde{C}^k_{HCE}(M) \xrightarrow{\tilde{\partial}_H} \tilde{C}^{k+1}_{HCE}(M) \longrightarrow \cdots$$

where $\tilde{\partial}_H = (\partial_H)_{|\tilde{C}^*_{HCE}(M)}$ and $\tilde{C}^*_{HCE}(M) = \oplus_k \tilde{C}^k_{HCE}(M)$. This subcomplex defines a cohomology which is denoted by $\tilde{H}^*_{HCE}(M)$. Using proposition 4.2, we deduce that the isomorphism $i : \mathcal{V}^k_I(M) \longrightarrow \tilde{C}^k_{HCE}(M)$ induces an isomorphism of complexes $i : (\mathcal{V}^*_I(M), \sigma_I) \longrightarrow (\tilde{C}^*_{HCE}(M), \tilde{\partial}_H)$. Therefore, we conclude with the following theorem.

Theorem 4.5. Let (M, Λ, E) be a Jacobi manifold. Then the kth LJ-cohomology group $H^k_{\mathrm{LJ}}(M)$ is isomorphic to the cohomology group $\tilde{H}^k_{HCE}(M)$.

5. Lichnerowicz–Jacobi cohomology and basic de Rham cohomology in a Jacobi manifold

In this section, we will obtain a homomorphism between the basic de Rham cohomology and the LJ-cohomology of a Jacobi manifold.

Let (M, Λ, E) be a Jacobi manifold. The mapping $\# : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ can be extended to a mapping, which we also denote by #, from the space of k-forms $\Omega^k(M)$ on M onto the space of k-vectors $\mathcal{V}^k(M)$ by putting:

$$\#(f) = f \qquad \#(\alpha)(\alpha_1, \ldots, \alpha_k) = (-1)^k \alpha(\#\alpha_1, \ldots, \#\alpha_k) \tag{18}$$

for $f \in C^\infty(M, \mathbb{R})$, $\alpha \in \Omega^k(M)$ and $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$.

We remark that if α and β are forms on M, then

$$\#(\alpha \wedge \beta) = \#\alpha \wedge \#\beta. \tag{19}$$

Denote by σ the differential operator given by (13).

Proposition 5.1. If (M, Λ, E) is a Jacobi manifold and α is a form on M then,

$$\mathcal{L}_E(\#\alpha) = \#(\mathcal{L}_E \alpha) \tag{20}$$
$$\sigma(\#\alpha) = -\#(\mathrm{d}\alpha) + \#(i_E \alpha) \wedge \Lambda. \tag{21}$$

Proof. Let α be a 1-form on M. Then, from (1) and (10), we have that

$$0 = (\mathcal{L}_E \Lambda)(\alpha, \beta) = \beta(\mathcal{L}_E(\#\alpha)) - \beta(\#(\mathcal{L}_E \alpha)) \tag{22}$$

for $\beta \in \Omega^1(M)$. Thus, $\mathcal{L}_E(\#\alpha) = \#(\mathcal{L}_E \alpha)$ which, using (19), implies (20).

Now, a straightforward computation proves that

$$[\Lambda, \#\alpha] = [\Lambda, i_\alpha \Lambda] = \#(\mathrm{d}\alpha) - \tfrac{1}{2} i_\alpha [\Lambda, \Lambda]$$

for $\alpha \in \Omega^1(M)$. Therefore, from (1) and (13), we obtain that

$$\sigma(\#\alpha) = -\#(\mathrm{d}\alpha) + \alpha(E)\Lambda. \tag{23}$$

Finally, using (16), (19) and (23), we deduce (21). □

A k-form α on a Jacobi manifold (M, Λ, E) is called *basic* if $i_E\alpha = 0$ and $\mathcal{L}_E\alpha = 0$.

Next, we consider the subcomplex of the de Rham complex given by the basic forms:

$$\cdots \longrightarrow \Omega_B^{k-1}(M) \xrightarrow{\mathrm{d}_B} \Omega_B^k(M) \xrightarrow{\mathrm{d}_B} \Omega_B^{k+1}(M) \longrightarrow \cdots$$

where $\Omega_B^k(M)$ is the space of basic k-forms, and $\mathrm{d}_B = \mathrm{d}_{|\Omega_B^*(M)}$. Its cohomology is denoted by $H_B^*(M)$ and called the *basic de Rham cohomology* of (M, Λ, E) (see [5, 7]).

From (20), we deduce that if $\alpha \in \Omega_B^k(M)$ then $\#\alpha \in \mathcal{V}_I^k(M)$. Consequently, the mapping $\# : \Omega^*(M) \longrightarrow \mathcal{V}^*(M)$ induces a homomorphism $\#_B : \Omega_B^*(M) \longrightarrow \mathcal{V}_I^*(M)$ of $C_B^\infty(M, \mathbb{R})$-modules. Moreover, from (21), we obtain that

$$\sigma_I \circ \#_B = -\#_B \circ \mathrm{d}_B \tag{24}$$

and, therefore, we deduce the following.

Theorem 5.2. Let (M, Λ, E) be a Jacobi manifold. Then, the mapping $\#_B$ induces a homomorphism of complexes $\#_B : (\Omega_B^*(M), \mathrm{d}_B) \longrightarrow (\mathcal{V}_I^*(M), -\sigma_I)$. Thus, we have the corresponding homomorphism in cohomology $\#_B : H_B^*(M) \longrightarrow H_{\mathrm{LJ}}^*(M)$.

Remark 5.3. For a Poisson manifold M, $H_B^*(M)$ is the de Rham cohomology of M and $\#_B = \#$. Consequently, using theorem 5.2, we obtain a homomorphism $\# : H_{\mathrm{dR}}^*(M) \longrightarrow H_{\mathrm{LP}}^*(M)$ between the de Rham cohomology of M and its LP-cohomology (see [21, 31]). In the particular case when M is a symplectic manifold then the homomorphism $\# : H_{\mathrm{dR}}^*(M) \longrightarrow H_{\mathrm{LP}}^*(M)$ is an isomorphism (see, for instance [31]).

Now, for a Jacobi manifold M, we consider the homomorphism of $C_B^\infty(M, \mathbb{R})$-modules $\tilde{\#}_B : \Omega_B^k(M) \longrightarrow \mathcal{V}_{IE}^{k+1}(M)$ given by

$$\tilde{\#}_B(\alpha) = E \wedge \#_B(\alpha). \tag{25}$$

From (16), (24) and (25), we obtain the following theorem.

Theorem 5.4. Let (M, Λ, E) be a Jacobi manifold. Then, the mapping $\tilde{\#}_B$ induces a homomorphism of complexes $\tilde{\#}_B : (\Omega_B^*(M), \mathrm{d}_B) \longrightarrow (\mathcal{V}_{IE}^{*+1}(M), \sigma_{IE})$. Thus, we have the corresponding homomorphism in cohomology $\tilde{\#}_B : H_B^*(M) \longrightarrow H_{\mathrm{LJE}}^{*+1}(M)$.

6. Natural pairing Lichnerowicz–Jacobi cohomology-canonical homology

Let us recall the notion of canonical homology for Jacobi manifolds introduced in [5, 7] which generalizes in a natural way the canonical homology for Poisson manifolds.

Let $\delta : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ be the differential operator given by

$$\delta = \mathrm{i}(\Lambda) \circ \mathrm{d} - \mathrm{d} \circ \mathrm{i}(\Lambda). \tag{26}$$

Note that, if M is a Poisson manifold, δ is just the *Koszul operator* (see [4, 17]).

Denote by δ_B the restriction of δ to $\Omega_B^k(M)$. It was proved in [5, 7] that $\delta_B(\Omega_B^k(M)) \subseteq \Omega_B^{k-1}(M)$ and $\delta_B^2 = 0$. These results allow us to introduce the differential complex

$$\cdots \longrightarrow \Omega_B^{k+1}(M) \xrightarrow{\delta_B} \Omega_B^k(M) \xrightarrow{\delta_B} \Omega_B^{k-1}(M) \longrightarrow \cdots$$

which is called the *canonical complex* of M. The homology of this complex is denoted by $H_*^{\mathrm{can}}(M)$, and it is called the *canonical homology* of M (we refer to [5, 7] for a more detailed study).

Now, consider the natural pairing $\langle\, ,\, \rangle : \Omega^k(M) \times \mathcal{V}^k(M) \longrightarrow C^\infty(M, \mathbb{R})$ defined by

$$\langle \alpha, P\rangle = \mathrm{i}(P)\alpha \tag{27}$$

where $\mathrm{i}(P)$ denotes the contraction by P.

Using (13), (26), (27) and the fact that $[[\mathrm{i}(P), \mathrm{d}], \mathrm{i}(Q)] = \mathrm{i}([P, Q])$ we deduce the following.

Proposition 6.1. If $\alpha \in \Omega^k_B(M)$ and $P \in \mathcal{V}^k_I(M)$, then $\langle \alpha, P\rangle \in C^\infty_B(M, \mathbb{R})$. Moreover,

$$\langle \alpha, \sigma_I(Q)\rangle - \langle \delta_B\alpha, Q\rangle = -\delta_B \mathrm{i}(Q)\alpha, \qquad \text{for } \alpha \in \Omega^k_B(M) \text{ and } Q \in \mathcal{V}^{k-1}_I(M).$$

From proposition 6.1 we obtain the following theorem.

Theorem 6.2. Let (M, Λ, E) be a Jacobi manifold. The mapping $\langle\, ,\, \rangle$ defined in (27) induces a natural pairing

$$\langle\, ,\, \rangle : H^{\mathrm{can}}_k(M) \times H^k_{\mathrm{LJ}}(M) \longrightarrow H^{\mathrm{can}}_0(M)$$

given by

$$\langle [\alpha], [P]\rangle = [\langle \alpha, P\rangle].$$

7. Quantizable Poisson manifolds and Lichnerowicz–Jacobi cohomology

In this section, we will study the relationship between the LP-cohomology of a quantizable Poisson manifold $\bar{M}$ and the LJ-cohomology of the total space of a prequantization bundle of $\bar{M}$. For this purpose, we will recall some definitions and results (see [6, 30, 31]).

As is well known, a one-to-one correspondence exists between the equivalence classes of principal circle bundles over a manifold $\bar{M}$ and the cohomology group $H^2(\bar{M}, \mathbb{Z})$. In fact, if $\bar{\Omega}$ is an integer closed 2-form on $\bar{M}$ then there exists a principal circle bundle $\pi : M \longrightarrow \bar{M}$ over $\bar{M}$ with connection form θ such that $\bar{\Omega}$ is the curvature for the connection θ, that is, $\pi^*\bar{\Omega} = \mathrm{d}\theta$ (see [15]).

Now, let $\pi : M \longrightarrow \bar{M}$ be a principal circle bundle over a manifold $\bar{M}$ endowed with a connection form θ. If $\bar{P}$ is a k-vector on $\bar{M}$, $k \geqslant 1$, we define the *horizontal lift* of $\bar{P}$ to M as the k-vector $\bar{P}^H$ on M characterized by the following conditions:

$$\bar{P}^H(\pi^*\bar{\alpha}_1, \ldots, \pi^*\bar{\alpha}_k) = \bar{P}(\bar{\alpha}_1, \ldots, \bar{\alpha}_k) \circ \pi \qquad i_\theta \bar{P}^H = 0 \tag{28}$$

for $\bar{\alpha}_1, \ldots, \bar{\alpha}_k \in \Omega^1(\bar{M})$. We remark that if $\bar{P} = \bar{X}_1 \wedge \ldots \wedge \bar{X}_k$ with $\bar{X}_i \in \mathfrak{X}(\bar{M})$ then

$$\bar{P}^H = \bar{X}^H_1 \wedge \ldots \wedge \bar{X}^H_k. \tag{29}$$

If $\bar{f} \in C^\infty(\bar{M}, \mathbb{R})$, the *horizontal lift* of $\bar{f}$ to M is the C^∞ real-valued function on M given by $\pi^*(\bar{f})$.

Next, suppose that $(\bar{M}, \bar{\Lambda})$ is a Poisson manifold.

We say that $\bar{M}$ is a *quantizable Poisson manifold* (see [30, 31]) if there exists a closed 2-form $\bar{\Omega}$ that represents an integral cohomology class of $\bar{M}$ such that

$$\bar{\#}[\bar{\Omega}] = [\bar{\Lambda}] \tag{30}$$

where $\bar{\#} : H^k_{\mathrm{dR}}(\bar{M}) \longrightarrow H^k_{\mathrm{LP}}(\bar{M})$ is the induced homomorphism (see remark 5.3). It is clear that (30) is equivalent to the existence of a vector field $\bar{A}$ on $\bar{M}$ and an integer closed 2-form $\bar{\Omega}$ on $\bar{M}$ such that

$$\bar{\Lambda} + \mathcal{L}_{\bar{A}}\bar{\Lambda} = \bar{\#}(\bar{\Omega}). \tag{31}$$

Remark 7.1. If $\bar{M}$ is a symplectic manifold with symplectic form $\bar{\omega}$ then, using (4), (18) and remark 2.1, we obtain that $\bar{\#}(\bar{\omega}) = \bar{\Lambda}$. Thus, from remark 5.3, we deduce that $\bar{M}$ is quantizable as a Poisson manifold if and only if $\bar{M}$ is quantizable as a symplectic manifold, that is (see [16]), if $[\bar{\omega}] \in H^2(\bar{M}, \mathbb{Z})$. Note that, in this case,

$$\bar{A} = 0 \qquad \text{and} \qquad \bar{\Omega} = \bar{\omega}. \tag{32}$$

Let $(\bar{M}, \bar{\Lambda})$ be a quantizable Poisson manifold and $\bar{A}$, $\bar{\Omega}$ a vector field and an integer closed 2-form on $\bar{M}$ satisfying (31). Consider the principal circle bundle $\pi : M \longrightarrow \bar{M}$ over $\bar{M}$ corresponding to $[\bar{\Omega}] \in H^2(\bar{M}, \mathbb{Z})$, which is called a *prequantization bundle* of $\bar{M}$. In [6] we proved that on M there exists a Jacobi structure (Λ, E) and a 1-form θ such that (M, Λ, E) is a regular Jacobi manifold and the corresponding quotient Poisson manifold M/E is just $(\bar{M}, \bar{\Lambda})$. Moreover,

$$\theta(E) = 1 \qquad \mathcal{L}_E\theta = 0. \tag{33}$$

In fact, θ is a connection form in $\pi : M \longrightarrow \bar{M}$ with curvature form $\bar{\Omega}$, E is the infinitesimal generator of the action of S^1 on M and Λ is given by

$$\Lambda = \bar{\Lambda}^H + E \wedge \bar{A}^H. \tag{34}$$

Conversely, a compact regular Jacobi manifold (M, Λ, E) endowed with a 1-form θ satisfying (33) is the total space of a prequantization bundle of a quantizable Poisson manifold (see [6]).

Remark 7.2. Let $(\bar{M}, \bar{\Lambda})$ be a quantizable symplectic manifold with symplectic 2-form $\bar{\Omega}$. We have that $[\bar{\Omega}] \in H^2(\bar{M}, \mathbb{Z})$ and that the principal circle bundle $\pi : M \longrightarrow \bar{M}$ over $\bar{M}$ corresponding to $[\bar{\Omega}]$ is a prequantization bundle of $\bar{M}$. Moreover, if θ is a connection form in $\pi : M \longrightarrow \bar{M}$ with curvature form $\bar{\Omega}$ then θ is a contact 1-form on M and the Jacobi structure on M defined by the contact 1-form θ is just the induced Jacobi structure on M by the quantizable symplectic manifold $(\bar{M}, \bar{\Omega})$ (see [6]).

Next, we will obtain the relationship between the LP-cohomology of a quantizable Poisson manifold $\bar{M}$ and the E-LJ-cohomology of the total space of a prequantization bundle of $\bar{M}$.

Theorem 7.3. Let $(\bar{M}, \bar{\Lambda})$ be a quantizable Poisson manifold which admits a prequantization bundle $\pi : M \longrightarrow \bar{M}$. Suppose that θ is a connection form in $\pi : M \longrightarrow \bar{M}$, and that (Λ, E) is the associated Jacobi structure on M. Denote by σ_I (respectively $\bar{\sigma}$) the LJ-cohomology (respectively LP-cohomology) operator on M (respectively $\bar{M}$). For every k, $0 \leqslant k \leqslant \dim \bar{M}$, we define the isomorphism $\bar{\phi}_k : \mathcal{V}^k(\bar{M}) \longrightarrow \mathcal{V}^{k+1}_{IE}(M)$ given by

$$\bar{\phi}_k(\bar{P}) = E \wedge \bar{P}^H. \tag{35}$$

Then, $\bar{\phi}_k$ induces an isomorphism of complexes $\bar{\phi} : (\mathcal{V}^*(\bar{M}), \bar{\sigma}) \longrightarrow (\mathcal{V}^{*+1}_{IE}(M), -\sigma_{IE})$, where $\sigma_{IE} = (\sigma_I)_{|\mathcal{V}^*_{IE}(M)}$. Therefore, the kth LP-cohomology group $H^k_{\mathrm{LP}}(\bar{M})$ of $\bar{M}$ is isomorphic to the $(k+1)$th E-LJ-cohomology group $H^{k+1}_{\mathrm{LJ}E}(M)$ of M.

Proof. Note that from (28), (33) and (35), we deduce that $\bar{\phi}_k$ is just an isomorphism. In fact, the inverse homomorphism is defined by

$$\bar{\phi}_k^{-1} : \mathcal{V}^{k+1}_{IE}(M) \longrightarrow \mathcal{V}^k(\bar{M}) \qquad P \longrightarrow \bar{P}$$

where $\bar{P}$ is the unique k-vector on $\bar{M}$ such that $\bar{P}^H = i_\theta P$.

On the other hand, using (29) and the properties of the Schouten–Nijenhuis bracket, we have that

$$[\bar{\Lambda}^H, \bar{P}^H] - E \wedge i_\theta[\bar{\Lambda}^H, \bar{P}^H] = [\bar{\Lambda}, \bar{P}]^H \tag{36}$$

for $\bar{P} \in \mathcal{V}^k(\bar{M})$. Finally, from (12), (13), (28) and (34)–(36), it follows that $\bar{\phi}_k(\bar{\sigma}(\bar{P})) = -\sigma_{IE}(\bar{\phi}_{k-1}(\bar{P}))$, for $\bar{P} \in \mathcal{V}^{k-1}(\bar{M})$. This completes the proof. □

Using theorem 7.3, we deduce that the homomorphisms of complexes $i : (\mathcal{V}^*_{IE}(M), \sigma_{IE}) \longrightarrow (\mathcal{V}^*_I(M), \sigma_I)$ and $\pi : (\mathcal{V}^*_I(M), \sigma_I) \longrightarrow (\mathcal{V}^{*+1}_{IE}(M), -\sigma_{IE})$ defined in proposition 3.3 induce a monomorphism of complexes $\tilde{i} : (\mathcal{V}^{*-1}(\bar{M}), -\bar{\sigma}) \longrightarrow (\mathcal{V}^*_I(M), \sigma_I)$ and an epimorphism of complexes $\tilde{\pi} : (\mathcal{V}^*_I(M), \sigma_I) \longrightarrow (\mathcal{V}^*(\bar{M}), \bar{\sigma})$, respectively. In fact, if $\bar{Q} \in \mathcal{V}^{k-1}(\bar{M})$ and $P \in \mathcal{V}^k_I(M)$, we get

$$\tilde{i}_k(\bar{Q}) = E \wedge \bar{Q}^H \qquad \tilde{\pi}_k(P) = \bar{P} \qquad \text{with } \bar{P}^H = P - E \wedge i_\theta P. \tag{37}$$

Now, we conclude the following.

Theorem 7.4. Let $(\bar{M}, \bar{\Lambda})$ be a quantizable Poisson manifold which admits a prequantization bundle $\pi : M \longrightarrow \bar{M}$. Suppose that σ_I (respectively $\bar{\sigma}$) is the LJ-cohomology (respectively LP-cohomology) operator on M (respectively $\bar{M}$). Then:

(i) there is an exact sequence of complexes

$$0 \longrightarrow (\mathcal{V}^{*-1}(\bar{M}), -\bar{\sigma}) \xrightarrow{\tilde{i}} (\mathcal{V}^*_I(M), \sigma_I) \xrightarrow{\tilde{\pi}} (\mathcal{V}^*(\bar{M}), \bar{\sigma}) \longrightarrow 0$$

(ii) this exact sequence induces a long exact cohomology sequence

$$\cdots \longrightarrow H^{k-1}_{\mathrm{LP}}(\bar{M}) \xrightarrow{\tilde{i}^*_k} H^k_{\mathrm{LJ}}(M) \xrightarrow{\tilde{\pi}^*_k} H^k_{\mathrm{LP}}(\bar{M}) \xrightarrow{\tilde{\Delta}^*_k} H^k_{\mathrm{LP}}(\bar{M}) \longrightarrow \cdots$$

with connecting homomorphism $\tilde{\Delta}^*$;

(iii) if the LP-cohomology groups of $\bar{M}$ have finite dimension then the LJ-cohomology groups of M also have finite dimension.

Next, in order to apply theorem 7.4 to the particular case of a quantizable symplectic manifold, we will give an explicit expression of the connecting homomorphism $\tilde{\Delta}^*_k$. For this purpose, we will prove some results.

Let $(\bar{M}, \bar{\Lambda})$ be a quantizable Poisson manifold which admits a prequantization bundle $\pi : M \longrightarrow \bar{M}$. Suppose that θ is a connection form in $\pi : M \longrightarrow \bar{M}$ with curvature form $\bar{\Omega}$ and that (Λ, E) is the associated Jacobi structure on M.

If $\bar{P} \in \mathcal{V}^k(\bar{M})$ then, from (28) and (33), we have that $\mathcal{L}_E(i_\theta[\bar{\Lambda}^H, \bar{P}^H]) = i_\theta(\mathcal{L}_E[\bar{\Lambda}^H, \bar{P}^H]) = 0$. Thus, we can define the homomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules $\mathcal{D}_\theta : \mathcal{V}^k(\bar{M}) \longrightarrow \mathcal{V}^k(\bar{M})$ by:

$$\mathcal{D}_\theta(\bar{P}) = \bar{Q} \qquad \text{with } \bar{Q}^H = i_\theta[\bar{\Lambda}^H, \bar{P}^H]. \tag{38}$$

Denote by $\bar{\flat} : \mathfrak{X}(\bar{M}) \longrightarrow \Omega^1(\bar{M})$ the homomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules given by

$$\bar{\flat}(\bar{X}) = i_{\bar{X}}\bar{\Omega} \tag{39}$$

for $\bar{X} \in \mathfrak{X}(\bar{M})$. We deduce the following.

Lemma 7.5. The homomorphism $\mathcal{D}_\theta$ defines a derivation of degree 0 of the algebra $\mathcal{V}^*(\bar{M})$ and $\mathcal{D}_\theta(\bar{f}) = 0$ and $\mathcal{D}_\theta(\bar{X}) = -\bar{\#}(\bar{\flat}(\bar{X}))$, for $\bar{f} \in C^\infty(\bar{M}, \mathbb{R})$ and $\bar{X} \in \mathfrak{X}(\bar{M})$.

Proof. Using (29), (38) and the properties of the Schouten–Nijenhuis bracket, we obtain that $\mathcal{D}_\theta$ is a derivation of degree 0 of the algebra $\mathcal{V}^*(\bar{M})$.

Now, if $\bar{f} \in C^\infty(\bar{M}, \mathbb{R})$ then, from (38), it follows that $\mathcal{D}_\theta(\bar{f}) = 0$.

On the other hand, if $\bar{X} \in \mathfrak{X}(\bar{M})$, using (28), (39) and the fact that $\mathrm{d}\theta = \pi^*(\bar{\Omega})$, we have that $\mathcal{L}_{\bar{X}^H}\theta = \pi^*(\bar{\flat}(\bar{X}))$. This implies that

$$i_\theta[\bar{\Lambda}^H, \bar{X}^H] = i_\theta(\mathcal{L}_{\bar{X}^H}\bar{\Lambda}^H) = -i_{(\mathcal{L}_{\bar{X}^H}\theta)}\bar{\Lambda}^H = -(\bar{\#}(\bar{\flat}(\bar{X})))^H.$$

Therefore, $\mathcal{D}_\theta(\bar{X}) = -(\bar{\#}(\bar{\flat}(\bar{X}))$. □

Remark 7.6. If $(\bar{M}, \bar{\Omega})$ is a quantizable symplectic manifold then, from (39), remark 2.1 and lemma 7.5, we deduce that $\mathcal{D}_\theta(\bar{P}) = k\bar{P}$, for $\bar{P} \in \mathcal{V}^k(\bar{M})$.

Let $\bar{P}$ be a k-vector on $\bar{M}$ such that $\bar{\sigma}(\bar{P}) = 0$.

From (12), (13), (28), (34) and (36), we obtain that

$$\sigma_I(\bar{P}^H) = E \wedge (k\bar{P}^H + (-1)^k(\mathcal{L}_{\bar{A}^H}\bar{P}^H) - (\mathcal{D}_\theta(\bar{P}))^H).$$

Thus, since $E \wedge (\mathcal{L}_{\bar{A}}\bar{P})^H = E \wedge (\mathcal{L}_{\bar{A}^H}\bar{P}^H)$ (see (28)), we have that (see (37))

$$\sigma_I(\bar{P}^H) = \tilde{i}_k(k\bar{P} + (-1)^k\mathcal{L}_{\bar{A}}\bar{P} - \mathcal{D}_\theta(\bar{P})). \tag{40}$$

Consequently, using (37), (40) and the definition of the connecting homomorphism $\tilde{\Delta}_k^*$, we conclude that

$$\tilde{\Delta}_k^*([\bar{P}]) = [k\bar{P} + (-1)^k\mathcal{L}_{\bar{A}}\bar{P} - \mathcal{D}_\theta(\bar{P})]. \tag{41}$$

Corollary 7.7. Let $(\bar{M}, \bar{\Omega})$ be a $2m$-dimensional quantizable symplectic manifold of finite type which admits a prequantization bundle $\pi : M \longrightarrow \bar{M}$. If θ is a connection form in $\pi : M \longrightarrow \bar{M}$ with curvature form $\bar{\Omega}$ then (M, θ) is a regular contact manifold and

$$H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{dR}}(\bar{M}) \oplus H^{k-1}_{\mathrm{dR}}(\bar{M}) \qquad H^{2m+1}_{\mathrm{LJ}}(M) \cong H^{2m}_{\mathrm{dR}}(\bar{M}) \cong \mathbb{R} \tag{42}$$

for $0 \leqslant k \leqslant 2m$, where $H^l_{\mathrm{dR}}(\bar{M})$ denotes the lth de Rham cohomology group of $\bar{M}$.

Proof. From (41) and remark 7.6, we deduce that the connecting homomorphism $\tilde{\Delta}_k^*$ vanishes (note that in this case $\bar{A} = 0$). Therefore, the result follows using remarks 5.3 and 7.2 and theorem 7.4. □

Next, we will apply corollary 7.7 to two particular cases: the two-dimensional unit sphere S^2 and the two-dimensional real torus $\mathbb{T}^2$.

Example 7.8. Let $i : S^2 \longrightarrow \mathbb{R}^3$ be the canonical inclusion and (x, y, z) the usual coordinates in $\mathbb{R}^3$. We consider on S^2 the symplectic 2-form $\bar{\Omega}$ defined by:

$$\bar{\Omega} = \frac{1}{4\pi} i^*(x\,\mathrm{d}y \wedge \mathrm{d}z - y\,\mathrm{d}x \wedge \mathrm{d}z + z\,\mathrm{d}x \wedge \mathrm{d}y).$$

$[\bar{\Omega}]$ is a generator of the integer cohomology group $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$. Consequently, the symplectic manifold $(S^2, \bar{\Omega})$ is quantizable.

As is well known, the special unitary group $SU(2)$ is the total space of the principal circle bundle over S^2 corresponding to the integer closed 2-form $\bar{\Omega}$. In fact, if we identify $SU(2)$ with the three-dimensional unit sphere S^3 via the diffeomorphism

$$(x^1, x^2, x^3, x^4) \in S^3 \mapsto \begin{pmatrix} x^1 + \mathrm{i}x^2 & -x^3 + \mathrm{i}x^4 \\ x^3 + \mathrm{i}x^4 & x^1 - \mathrm{i}x^2 \end{pmatrix} \in SU(2) \tag{43}$$

then the projection of the bundle is the *Hopf fibration* $\pi : SU(2) \cong S^3 \longrightarrow S^2$ and the action of S^1 on S^3 is the usual one (see [6]).

Now, we will describe the contact and Jacobi structures induced on $SU(2)$. Denote by σ_1, σ_2 and σ_3 the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, the matrices $\{\sqrt{\pi}\mathrm{i}\sigma_1, \sqrt{\pi}\mathrm{i}\sigma_2, 2\pi\mathrm{i}\sigma_3\}$ form a basis of the Lie algebra of $SU(2)$ which defines on $SU(2)$ a basis of left invariant vector fields $\{X, Y, \xi\}$. Moreover, if $\{\alpha, \beta, \theta\}$ is the dual basis of left invariant 1-forms, we have that θ is a connection form in the principal circle bundle $\pi : SU(2) \simeq S^3 \longrightarrow S^2$ with curvature form $\bar{\Omega}$ (see [6]). Thus, θ is a contact 1-form on $SU(2)$ and the Jacobi structure (Λ, E) defined by θ on $SU(2)$ is just the Jacobi structure on $SU(2)$ induced by the quantizable symplectic manifold $(S^2, \bar{\Omega})$. In fact (see [6]),

$$\Lambda = X \wedge Y \qquad E = \xi.$$

Finally, from corollary 7.7, we conclude that

$$H^0_{\mathrm{LJ}}(SU(2)) = \langle [1] \rangle \cong \mathbb{R} \qquad H^1_{\mathrm{LJ}}(SU(2)) = \langle [E] \rangle \cong \mathbb{R}$$
$$H^2_{\mathrm{LJ}}(SU(2)) = \langle [\Lambda] \rangle \cong \mathbb{R} \qquad H^3_{\mathrm{LJ}}(SU(2)) = \langle [E \wedge \Lambda] \rangle \cong \mathbb{R}.$$

Example 7.9. Let $\tilde{\Omega}$ be the usual symplectic 2-form on $\mathbb{R}^2$, $\tilde{\Omega} = \mathrm{d}q \wedge \mathrm{d}p$, (q, p) being the canonical coordinates on $\mathbb{R}^2$. Denote by $\bar{\Omega}$ the symplectic 2-form on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by $\tilde{\Omega}$. $[\bar{\Omega}]$ is a generator of the integer cohomology group $H^2(\mathbb{T}^2, \mathbb{Z}) = \mathbb{Z}$. Thus, the symplectic manifold $(\mathbb{T}^2, \bar{\Omega})$ is quantizable.

Now, let H be the *Heisenberg group*. It is well known that H is the Lie group of matrices of real numbers of the form

$$A = \begin{pmatrix} 1 & q & t \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix}$$

with $q, p, t \in \mathbb{R}$. H is a simply connected nilpotent three-dimensional Lie group. A global system of coordinates (q, p, t) on H is defined by $q(A) = q$, $p(A) = p$, $t(A) = t$.

A basis of right invariant vector fields on H is given by

$$\tilde{X} = \frac{\partial}{\partial q} + p\frac{\partial}{\partial t} \qquad \tilde{Y} = \frac{\partial}{\partial p} \qquad \tilde{\xi} = \frac{\partial}{\partial t}.$$

Denote by Γ the subgroup of matrices of H with integer entries and by $M = H/\Gamma$ the space of left cosets; M is a compact nilmanifold. The vector fields $\tilde{X}, \tilde{Y}, \tilde{\xi}$ on H all descend to M; denote the vector fields induced on M by X, Y and ξ, respectively.

The space M is a principal circle bundle over $\mathbb{T}^2$. The projection $\pi : M \longrightarrow \mathbb{T}^2$ of the bundle is defined by $\pi[(q, p, t)] = [(q, p)]$. The infinitesimal generator of the action of S^1 on M is the vector field ξ. Moreover, if $\tilde{\theta}$ is the right-invariant 1-form on H given by $\tilde{\theta} = \mathrm{d}t - p\,\mathrm{d}q$, then $\tilde{\theta}$ induces a 1-form θ on M and θ is a connection form in the principal circle bundle $\pi : M \longrightarrow \mathbb{T}^2$ with curvature form $\bar{\Omega}$ (see [6]).

Therefore, the Jacobi structure (Λ, E) defined on M by the contact 1-form θ is just the Jacobi structure induced on M by the quantizable symplectic manifold $(\mathbb{T}^2, \bar{\Omega})$. In fact, we have that

$$\Lambda = X \wedge Y \qquad E = \xi.$$

Finally, from corollary 7.7, we conclude that

$$H^0_{\mathrm{LJ}}(M) = \langle [1] \rangle \cong \mathbb{R} \qquad H^1_{\mathrm{LJ}}(M) = \langle [X], [Y], [E] \rangle \cong \mathbb{R}^3$$
$$H^2_{\mathrm{LJ}}(M) = \langle [\Lambda], [E \wedge X], [E \wedge Y] \rangle \cong \mathbb{R}^3 \qquad H^3_{\mathrm{LJ}}(M) = \langle [E \wedge \Lambda] \rangle \cong \mathbb{R}.$$

8. Lichnerowicz–Jacobi cohomology of cosymplectic manifolds

In this section, we will study the LJ-cohomology of a cosymplectic manifold. Note that, since a cosymplectic manifold is a Poisson manifold, the LJ-cohomology is just the LP-cohomology.

Let $(\bar{M}, \bar{\Phi}, \bar{\eta})$ be a cosymplectic manifold. Denote by $\bar{\flat} : \mathfrak{X}(\bar{M}) \longrightarrow \Omega^1(\bar{M})$ the isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules given by $\bar{\flat}(\bar{X}) = i_{\bar{X}}\bar{\Phi} + \bar{\eta}(\bar{X})\bar{\eta}$. Then, $\bar{\flat}$ can be extended to a mapping, also denoted by $\bar{\flat}$, from the space $\mathcal{V}^k(\bar{M})$ onto the space $\Omega^k(\bar{M})$ by putting:

$$\bar{\flat}(\bar{X}_1 \wedge \ldots \wedge \bar{X}_k) = \bar{\flat}(\bar{X}_1) \wedge \ldots \wedge \bar{\flat}(\bar{X}_k) \tag{44}$$

for $\bar{X}_1, \ldots, \bar{X}_k \in \mathfrak{X}(\bar{M})$. Thus, $\bar{\flat}$ is also an isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules. Moreover, if $\bar{\#} : \Omega^k(\bar{M}) \longrightarrow \mathcal{V}^k(\bar{M})$ is the mapping defined in section 5 (see (18)) then, using (19), (44) and remark 2.1, we deduce the following.

Proposition 8.1. Let $(\bar{M}, \bar{\Phi}, \bar{\eta})$ be a cosymplectic manifold with Reeb vector field $\bar{\xi}$. For every $\bar{\alpha} \in \Omega^k(\bar{M})$ we have that

$$\bar{\#}\bar{\alpha} = (-1)^k \bar{\flat}^{-1}(\bar{\alpha}) + \bar{\xi} \wedge \bar{\#}(i_{\bar{\xi}}\bar{\alpha}).$$

Now, we consider the submodule $\Omega^k_{\bar{\xi}}(\bar{M})$ of $\Omega^k(\bar{M})$ given by $\Omega^k_{\bar{\xi}}(\bar{M}) = \{\bar{\alpha} \in \Omega^k(\bar{M}) / i_{\bar{\xi}}\bar{\alpha} = 0\}$, and define the operator $\mathrm{d}_{\bar{\xi}} : \Omega^k_{\bar{\xi}}(\bar{M}) \longrightarrow \Omega^{k+1}_{\bar{\xi}}(\bar{M})$ by

$$\mathrm{d}_{\bar{\xi}}\bar{\alpha} = \mathrm{d}\bar{\alpha} - \bar{\eta} \wedge i_{\bar{\xi}}\, \mathrm{d}\bar{\alpha} \qquad \text{for } \bar{\alpha} \in \Omega^k_{\bar{\xi}}(\bar{M}). \tag{45}$$

We have that $\mathrm{d}^2_{\bar{\xi}} = 0$ (see [9]) and thus we can consider the corresponding differential complex $(\Omega^*_{\bar{\xi}}(\bar{M}), \mathrm{d}_{\bar{\xi}})$. We denote by $H^*_{\bar{\xi}}(\bar{M})$ the cohomology of this complex.

Let $\bar{\sigma}$ denote the LP-cohomology operator on $\bar{M}$. We obtain the following.

Theorem 8.2. Let $(\bar{M}, \bar{\Phi}, \bar{\eta})$ be a cosymplectic manifold with Reeb vector field $\bar{\xi}$. Suppose that $\bar{F}_k : \Omega^k_{\bar{\xi}}(\bar{M}) \oplus \Omega^{k-1}_{\bar{\xi}}(\bar{M}) \longrightarrow \mathcal{V}^k(\bar{M})$ is the homomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules given by

$$\bar{F}_k(\bar{\alpha}, \bar{\beta}) = \bar{\#}\bar{\alpha} + \bar{\xi} \wedge \bar{\#}\bar{\beta}.$$

(i) $\bar{F}_k$ induces an isomorphism of complexes $\bar{F} : (\Omega^*_{\bar{\xi}}(\bar{M}), -\mathrm{d}_{\bar{\xi}}) \oplus (\Omega^{*-1}_{\bar{\xi}}(\bar{M}), \mathrm{d}_{\bar{\xi}}) \longrightarrow (\mathcal{V}^*(\bar{M}), \bar{\sigma})$.

(ii) For every k, $0 \leqslant k \leqslant \dim \bar{M}$, we have that $H^k_{\mathrm{LP}}(\bar{M}) \cong H^k_{\bar{\xi}}(\bar{M}) \oplus H^{k-1}_{\bar{\xi}}(\bar{M})$.

Proof. Since $\bar{\eta}$ is closed, $\mathcal{L}_{\bar{\xi}}\bar{\Phi} = 0$ and $\mathcal{L}_{\bar{\xi}}\bar{\eta} = 0$, we have that $\mathcal{L}_{\bar{\xi}}(\bar{\flat}^{-1}(\bar{\alpha})) = \bar{\flat}^{-1}(\mathcal{L}_{\bar{\xi}}\bar{\alpha})$, for $\bar{\alpha} \in \Omega^k(\bar{M})$.Thus, using (5), we prove that

$$\bar{\sigma}(\bar{\xi}) = 0. \tag{46}$$

From (16), (45), (46), the results of section 5 and since $\bar{\#}\bar{\eta} = 0$, we conclude that $\bar{F}$ is a homomorphism of complexes. Moreover, using (19), (44) and proposition 8.1, we obtain that $\bar{F}_k$ is an isomorphism of $C^\infty(\bar{M}, \mathbb{R})$-modules. In fact, the inverse homomorphism is defined by

$$\bar{P} \in \mathcal{V}^k(\bar{M}) \longrightarrow \bar{F}_k^{-1}(\bar{P}) = ((-1)^k(\bar{\flat}(\bar{P}) - \bar{\eta} \wedge i_{\bar{\xi}}\bar{\flat}(\bar{P})), (-1)^{k-1} i_{\bar{\xi}}\bar{\flat}(\bar{P}))$$
$$\in \Omega^k_{\bar{\xi}}(\bar{M}) \oplus \Omega^{k-1}_{\bar{\xi}}(\bar{M}).$$

□

In [9] (see also [10]) the authors have proved that if $\bar{M}$ is a $(2m+1)$-dimensional cosymplectic manifold then

$$H^{\mathrm{can}}_k(\bar{M}) \cong H^{2m+1-k}_{\bar{\xi}}(\bar{M}) \oplus H^{2m-k}_{\bar{\xi}}(\bar{M}) \tag{47}$$

where $H^{\mathrm{can}}_k(\bar{M})$ is the kth canonical homology group of $\bar{M}$.

From (47) and theorem 8.2, we obtain the following.

Corollary 8.3. Let $\bar{M}$ be a $(2m+1)$-dimensional cosymplectic manifold. Then the kth canonical homology group $H_k^{\mathrm{can}}(\bar{M})$ of $\bar{M}$ is isomorphic to the $(2m+1-k)$th LP-cohomology group $H_{\mathrm{LP}}^{2m+1-k}(\bar{M})$, for all k.

It is clear that the LP-cohomology groups of a compact symplectic manifold have finite dimension (see remark 5.3). Using theorem 8.2, we will prove that the corresponding result does not hold for compact cosymplectic manifolds. In fact, we will construct a counterexample.

Example 8.4. Let $\bar{N}$ be a compact symplectic manifold with symplectic 2-form $\bar{\Omega}$. Consider the following cosymplectic structure $(\bar{\Phi}, \bar{\eta})$ on $\bar{M} = \bar{N} \times S^1$:

$$\bar{\Phi} = (pr_1)^*(\bar{\Omega}) \qquad \bar{\eta} = (pr_2)^*(\theta) \tag{48}$$

where pr_1 and pr_2 are the canonical projections of $\bar{M}$ onto the first and second factor, respectively, and θ is the length element of S^1. Note that the Reeb vector field $\bar{\xi}$ of $\bar{M}$ is the vector field $\bar{\xi}$ on S^1 characterized by the condition $\theta(\bar{\xi}) = 1$.

Denote by $H_{\mathrm{dR}}^*(\bar{N})$ the de Rham cohomology of $\bar{N}$, and consider the $\mathbb{R}$-bilinear mapping

$$H_{\mathrm{dR}}^k(\bar{N}) \times C^\infty(S^1, \mathbb{R}) \longrightarrow H_{\bar{\xi}}^k(\bar{M}) \qquad ([\bar{\alpha}], f) \longrightarrow [pr_2^*(f)\, pr_1^*(\bar{\alpha})].$$

Since $H_{\mathrm{dR}}^k(\bar{N})$ has finite dimension we deduce that the above mapping induces an isomorphism between the real vector spaces $H_{\mathrm{dR}}^k(\bar{N}) \otimes C^\infty(S^1, \mathbb{R})$ and $H_{\bar{\xi}}^k(\bar{M})$. In particular, $H_{\bar{\xi}}^k(\bar{M})$ has infinite dimension. Thus, the LP-cohomology groups of $\bar{M}$ have also infinite dimension. In fact, using theorem 8.2, we conclude that

$$H_{\mathrm{LP}}^k(\bar{M}) \cong (H_{\mathrm{dR}}^k(\bar{N}) \otimes C^\infty(S^1, \mathbb{R})) \oplus (H_{\mathrm{dR}}^{k-1}(\bar{N}) \otimes C^\infty(S^1, \mathbb{R}))$$

for $0 \leqslant k \leqslant \dim \bar{M}$.

Remark 8.5.

(i) It is clear that if $[\bar{\Omega}] \in H^2(\bar{N}, \mathbb{Z})$ then $[\bar{\Phi}] \in H^2(\bar{M}, \mathbb{Z})$.

(ii) Let $(\bar{M}, \bar{\Phi}, \bar{\eta})$ be a compact cosymplectic manifold such that $[\bar{\Phi}] \in H^2(\bar{M}, \mathbb{Z})$. Suppose that $\pi : M \longrightarrow \bar{M}$ is the principal circle bundle over $\bar{M}$ corresponding to $[\bar{\Phi}]$. Then, $\bar{M}$ is a quantizable Poisson manifold and $\pi : M \longrightarrow \bar{M}$ is a prequantization bundle of $\bar{M}$ (see [6]). Moreover, there exists a l.c.s. structure on M in such a sense that M is a regular l.c.s. manifold and the corresponding quotient Poisson manifold is just $\bar{M}$ (see [6] and example 10.3). Using the results of section 10 (see remark 10.5 and corollary 10.9), we can prove that the LJ-cohomology groups of M have finite dimension. However, in general, the LP-cohomology groups of $\bar{M}$ do not have finite dimension (see example 8.4).

9. Lichnerowicz–Jacobi cohomology of contact manifolds

In this section, we will study the LJ-cohomology of a contact manifold.

Let (M, θ) be a contact manifold. Denote by $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X(\mathrm{d}\theta) + \theta(X)\theta$. The isomorphism $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ can be extended to a mapping, which we also denote by $\flat$, from the space $\mathcal{V}^k(M)$ onto the space $\Omega^k(M)$ by putting:

$$\flat(X_1 \wedge \ldots \wedge X_k) = \flat(X_1) \wedge \ldots \wedge \flat(X_k) \tag{49}$$

for $X_1, \ldots, X_k \in \mathfrak{X}(M)$. Thus, $\flat$ is also an isomorphism of $C^\infty(M, \mathbb{R})$-modules. Moreover, if $\# : \Omega^k(M) \longrightarrow \mathcal{V}^k(M)$ is the mapping defined in section 5 (see (18)) then, using (19), (49) and remark 2.1, we deduce the following.

Proposition 9.1. Let (M,θ) be a contact manifold and (Λ, E) the associated Jacobi structure on M. For every $\alpha \in \Omega^k(M)$, we have

$$\#\alpha = (-1)^k \ \flat^{-1}(\alpha) + E \wedge \#(i_E\alpha).$$

If $\#_B$ is the restriction of $\#$ to the basic forms and σ_I is the LJ-cohomology operator, we obtain the following.

Theorem 9.2. Let (M,θ) be a contact manifold and (Λ, E) the associated Jacobi structure on M. Suppose that $F_k : \Omega_B^k(M)\oplus\Omega_B^{k-1}(M) \longrightarrow \mathcal{V}_I^k(M)$ is the homomorphism of $C_B^\infty(M,\mathbb{R})$-modules given by

$$F_k(\alpha,\beta) = \#_B(\alpha) + E \wedge \#_B(\beta).$$

(i) F_k induces an isomorphism of complexes $F : (\Omega_B^*(M), -\mathrm{d}_B) \oplus (\Omega_B^{*-1}(M), \mathrm{d}_B) \longrightarrow (\mathcal{V}_I^*(M), \sigma_I)$.

(ii) For every k, $0 \leqslant k \leqslant \dim M$, we have that $H_{\mathrm{LJ}}^k(M) \cong H_B^k(M) \oplus H_B^{k-1}(M)$.

Proof. Using (16), the results of section 5 and the fact that $\sigma_I(E) = 0$, we conclude that F is a homomorphism of complexes.

On the other hand, from (49) and since $i_E\theta = 1$ and $i_E\,\mathrm{d}\theta = 0$, it follows that $\mathcal{L}_E \circ \flat = \flat \circ \mathcal{L}_E$. Therefore, we can define the homomorphism of $C_B^\infty(M,\mathbb{R})$-modules $G_k : \mathcal{V}_I^k(M) \longrightarrow \Omega_B^k(M) \oplus \Omega_B^{k-1}(M)$ by

$$G_k(P) = ((-1)^k(\flat(P) - \theta \wedge i_E\flat(P)), (-1)^{k-1} i_E\flat(P)).$$

Finally, using (19), (49) and proposition 9.1, we obtain that G_k is the inverse homomorphism of F_k. □

Remark 9.3.

(i) Equation (42) follows directly from theorem 9.2.

(ii) Let (M,θ) be a contact manifold and $\tilde{\#}_B : (\Omega_B^*(M), \mathrm{d}_B) \longrightarrow (\mathcal{V}_{IE}^{*+1}(M), \sigma_{IE})$ the homomorphism of complexes defined in section 5 (see (25) and theorem 5.4). We consider the homomorphism of $C_B^\infty(M,\mathbb{R})$-modules $H_k : \mathcal{V}_{IE}^{k+1}(M) \longrightarrow \Omega_B^k(M)$ given by

$$H_k(P) = (-1)^k\flat(i_\theta(P)) \tag{50}$$

for $P \in \mathcal{V}_{IE}^{k+1}(M)$. Note that $i_E\circ\flat = \flat\circ i_\theta$. Furthermore, using (25), (50) and proposition 9.1, we have that $\tilde{\#}_B(H_k(P)) = P$ and $H_k(\tilde{\#}_B(\alpha)) = \alpha$, for $P \in \mathcal{V}_{IE}^{k+1}(M)$ and $\alpha \in \Omega_B^k(M)$. Therefore, in this case, $\tilde{\#}_B : (\Omega_B^*(M), \mathrm{d}_B) \longrightarrow (\mathcal{V}_{IE}^{*+1}(M), \sigma_{IE})$ is an isomorphism of complexes and $H_B^k(M) \cong H_{\mathrm{LJ}E}^{k+1}(M)$.

In [5] (see also [7]) the authors have proved that if M is a $(2m+1)$-dimensional contact manifold, the canonical homology group $H_k^{\mathrm{can}}(M)$ is isomorphic to the basic de Rham cohomology group $H_B^{2m-k}(M)$. Using this result, theorem 9.2 and remark 9.3, we conclude the following.

Corollary 9.4. Let M be a $(2m+1)$-dimensional contact manifold with associated Jacobi structure (Λ, E). Then, the kth E-LJ-cohomology group $H_{\mathrm{LJ}E}^k(M)$ is isomorphic to the $(2m+1-k)$th canonical homology group $H_{2m+1-k}^{\mathrm{can}}(M)$. Thus, there exists also the following isomorphism:

$$H_{\mathrm{LJ}}^k(M) \cong H_{2m+1-k}^{\mathrm{can}}(M) \oplus H_{2m-k}^{\mathrm{can}}(M).$$

Theorem 9.2 also allows us to obtain sufficient conditions to ensure the finiteness of the LJ-cohomology groups of a particular class of compact contact manifolds. In fact, using theorem 9.2 and the results of [1] (see also [28, theorem 10.13, p 139]), we deduce the following.

Corollary 9.5. Let M be a compact contact manifold with associated Jacobi structure (Λ, E). Suppose that there exists a Riemannian metric g on M such that E is Killing with respect to g. Then the LJ-cohomology groups have finite dimension.

Next, we will compute the LJ-cohomology of a non-regular compact contact manifold on which there exists a Riemannian metric such that the Reeb vector field is Killing with respect to it.

Note that, in this case, it is not possible to use the results of section 7 to compute the LJ-cohomology.

Example 9.6. Using the identification of the three-dimensional unit sphere S^3 with $SU(2)$ given by (43) we will adapt a construction by Tanno [26] (see also [3, pp 90, 91]).

Denote by θ the contact 1-form on $SU(2)$ in example 7.8, and by (Λ, E) the associated Jacobi structure. Let g be the Riemannian metric on $SU(2) \cong S^3$ induced from the flat Riemannian metric of $\mathbb{R}^4$. It is well known that the Reeb vector field E is Killing with respect to g (see, for instance, [3]). Moreover, with an adequate homothetic transformation of the usual Riemannian metric of constant curvature 1 on S^2, we have that the Hopf fibration $\pi : SU(2) \cong S^3 \longrightarrow S^2$ is a Riemannian submersion. On the other hand, a direct computation using (43) proves that g is a left invariant metric on $SU(2)$.

Now, suppose that $a \in SU(2)$ satisfies $a^s = 1$ for some integer $s > 1$, and denote by ρ the left translation by a. Since ρ preserves the contact 1-form θ, the vector field E and the metric g, we deduce that ρ induces an automorphism $\bar{\rho}$ of the standard Kähler structure of S^2 (see [27]).

Let Γ be the finite cyclic subgroup of $SU(2)$ generated by a. Then, the space of right cosets $M = \Gamma \setminus SU(2)$ is a compact manifold. Furthermore, the 1-form θ and the Riemannian metric g induce a contact 1-form $\tilde{\theta}$ and a Riemannian metric $\tilde{g}$ on M such that the projected vector field $\tilde{E}$ is Killing with respect to $\tilde{g}$. It is obvious that $\tilde{E}$ is the Reeb vector field of the contact manifold $(M, \tilde{\theta})$.

It is not hard to choose $a \in SU(2)$ such that the induced automorphism $\bar{\rho}$ is non-trivial. Thus, using the fact that $\bar{\rho}$ has fixed points, we deduce that the orbits of E over such fixed points are invariant by ρ. Therefore, the period function of $\tilde{E}$ on M is not constant, and the contact manifold $(M, \tilde{\theta})$ is not regular.

Next, we will compute the basic de Rham cohomology of $(M, \tilde{\theta})$.

It is clear that $H^1_{\mathrm{dR}}(M) = \{0\}$ (note that the fundamental group of M is the finite cyclic group Γ). Consequently, using the results of [28, p 119]) we have that

$$H^0_B(M) = \langle [1] \rangle \cong \mathbb{R} \qquad H^1_B(M) = \{0\}. \tag{51}$$

Now, from theorem 9.23 of [28], we deduce that the cohomology class $[\mathrm{d}\tilde{\theta}] \in H^2_B(M)$ is nontrivial. Thus, since $H^2_B(M) \cong \mathbb{R}$ (see theorem 10.17 of [28]) it follows that

$$H^2_B(M) = \langle [\mathrm{d}\tilde{\theta}] \rangle \cong \mathbb{R}. \tag{52}$$

Finally, using (51), (52) and theorem 9.2, we conclude that

$$\begin{aligned} &H^0_{\mathrm{LJ}}(M) = \langle [1] \rangle \cong \mathbb{R} \qquad H^1_{\mathrm{LJ}}(M) = \langle [\tilde{E}] \rangle \cong \mathbb{R} \\ &H^2_{\mathrm{LJ}}(M) = \langle [\tilde{\Lambda}] \rangle \cong \mathbb{R} \qquad H^3_{\mathrm{LJ}}(M) = \langle [\tilde{E} \wedge \tilde{\Lambda}] \rangle \cong \mathbb{R} \end{aligned}$$

where $(\tilde{\Lambda}, \tilde{E})$ is the Jacobi structure associated to the contact manifold $(M, \tilde{\theta})$.

10. Lichnerowicz–Jacobi cohomology of locally conformal symplectic manifolds

In this section we shall study the LJ-cohomology of a l.c.s. manifold.

Let (M, Ω) be a l.c.s. manifold with Lee 1-form ω. We consider the differential operator $\mathrm{d}_\omega : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ given by

$$\mathrm{d}_\omega(\alpha) = \mathrm{d}\alpha + \omega \wedge \alpha \tag{53}$$

for $\alpha \in \Omega^k(M)$. We deduce that $\mathrm{d}_\omega^2 = 0$. Thus, we have the corresponding differential complex $(\Omega^*(M), \mathrm{d}_\omega)$. Denote by $H^*_\omega(M)$ the cohomology of this complex.

Proposition 10.1. Let M be a l.c.s. manifold with Lee 1-form ω. Then:

(i) The differential complex $(\Omega^*(M), \mathrm{d}_\omega)$ is elliptic. Therefore, if M is compact the cohomology groups $H^k_\omega(M)$ have finite dimension.

(ii) If M is a g.c.s. manifold then $H^k_{\mathrm{dR}}(M) \cong H^k_\omega(M)$.

Proof.

(i) A direct computation proves that the differential operators d and d_ω have the same symbol. Thus, the result follows.

(ii) Assume that $\omega = \mathrm{d}f$ with $f \in C^\infty(M, \mathbb{R})$. Using (53), we obtain that the mapping

$$H^k_{\mathrm{dR}}(M) \longrightarrow H^k_\omega(M) \qquad [\alpha] \longrightarrow [e^{-f}\alpha]$$

is an isomorphism. □

Now, suppose that (Λ, E) is the associated Jacobi structure on a l.c.s. manifold (M, Ω) with Lee 1-form ω. From (9) and (53), we deduce that

$$i_E \circ \mathrm{d}_\omega + \mathrm{d}_\omega \circ i_E = \mathcal{L}_E \qquad \text{and} \qquad \mathcal{L}_E \circ \mathrm{d}_\omega = \mathrm{d}_\omega \circ \mathcal{L}_E. \tag{54}$$

These results allow us to introduce the subcomplex of the complex $(\Omega^*(M), \mathrm{d}_\omega)$ given by:

$$\cdots \longrightarrow \Omega^{k-1}_B(M) \xrightarrow{\mathrm{d}_{\omega B}} \Omega^k_B(M) \xrightarrow{\mathrm{d}_{\omega B}} \Omega^{k+1}(M) \longrightarrow \cdots$$

where $\mathrm{d}_{\omega B} = (\mathrm{d}_\omega)_{|\Omega^*_B(M)}$. Denote by $H^*_{\omega B}(M)$ the cohomology of $(\Omega^*_B(M), \mathrm{d}_{\omega B})$.

Next, we will obtain sufficient conditions which ensure the finiteness of the cohomology groups $H^k_{\omega B}(M)$.

Using (9) and proceeding as in the proof of proposition 10.1, we have the following.

Proposition 10.2. Let M be a g.c.s. manifold with Lee 1-form ω. Then, $H^k_{\omega B}(M) \cong H^k_B(M)$. In particular, if $H^k_B(M)$ has finite dimension then $H^k_{\omega B}(M)$ also has finite dimension.

Example 10.3. Let $(\bar{M}, \bar{\Phi}, \bar{\eta})$ be a cosymplectic manifold with Reeb vector field $\bar{\xi}$. We define the differential operator $\mathrm{d}_{\bar{\eta}} : \Omega^k(\bar{M}) \longrightarrow \Omega^{k+1}(\bar{M})$ by

$$\mathrm{d}_{\bar{\eta}}(\bar{\alpha}) = \mathrm{d}\bar{\alpha} + \bar{\eta} \wedge \bar{\alpha} \tag{55}$$

for $\alpha \in \Omega^k(\bar{M})$. Since $\mathrm{d}_{\bar{\eta}}^2 = 0$, we can consider the corresponding differential complex $(\Omega^*(\bar{M}), \mathrm{d}_{\bar{\eta}})$ whose cohomology groups are denoted by $H^*_{\bar{\eta}}(\bar{M})$.

Now, assume that $[\bar{\Phi}] \in H^2(\bar{M}, \mathbb{Z})$ and let $\pi : M \longrightarrow \bar{M}$ be the principal circle bundle over $\bar{M}$ corresponding to $[\bar{\Phi}]$. If θ is a connection form in $\pi : M \longrightarrow \bar{M}$ with curvature form $\bar{\Phi}$, we define on M the 2-form Ω given by

$$\Omega = \mathrm{d}\theta - (\pi^*\bar{\eta}) \wedge \theta = \pi^*\bar{\Phi} - (\pi^*\bar{\eta}) \wedge \theta. \tag{56}$$

Then, it can be proved (see [6, 7]) that (M, Ω) is a regular l.c.s. manifold with Lee 1-form

$$\omega = \pi^*\bar{\eta}. \tag{57}$$

The corresponding quotient Poisson manifold is $(\bar{M}, \bar{\Phi}, \bar{\eta})$.

On the other hand, it is clear that the isomorphism $\pi^* : \Omega^k(\bar{M}) \longrightarrow \Omega^k_B(M)$ satisfies $\mathrm{d}_{\omega B} \circ \pi^* = \pi^* \circ \mathrm{d}_{\bar{\eta}}$ (see (53), (55) and (57)). Therefore, $H^k_{\omega B}(M) \cong H^k_{\bar{\eta}}(\bar{M})$. In particular, if the dimension of $H^k_{\bar{\eta}}(\bar{M})$ is finite then the dimension of $H^k_{\omega B}(M)$ is also finite.

Finally, we will consider a particular case. Suppose that $(\bar{N}, \bar{\Omega})$ is a quantizable symplectic manifold and that $\bar{M}$ is the product manifold $\bar{N} \times \mathbb{R}$ with the cosymplectic structure $(\bar{\Phi}, \bar{\eta})$ given by

$$\bar{\Phi} = pr_1^*(\bar{\Omega}) \qquad \text{and} \qquad \bar{\eta} = pr_2^*(\mathrm{d}t) \tag{58}$$

where t is the usual coordinate on $\mathbb{R}$ and pr_1, pr_2 are the canonical projections of $\bar{M}$ onto the first and second factor, respectively. In this case, (M, Ω) is a g.c.s. manifold and, from proposition 10.2, we have that

$$H^k_{\omega B}(M) \cong H^k_{\bar{\eta}}(\bar{M}) \cong H^k_{\mathrm{dR}}(\bar{M}) \cong H^k_{\mathrm{dR}}(\bar{N}). \tag{59}$$

Let (M, Ω) be a compact l.c.s. manifold with associated Jacobi structure (Λ, E). If $E \neq 0$ at every point and there exists a Riemannian metric g on M such that E is Killing with respect to g then the cohomology groups $H^k_B(M)$ have finite dimension (see [1, 28]). We will show that these conditions also ensure the finiteness of the cohomology groups $H^k_{\omega B}(M)$. We remark that, under the above conditions, M cannot be a g.c.s. manifold (note that a C^∞ real-valued function on a compact manifold always has at least two critical points).

Theorem 10.4. Let M be a $2m$-dimensional compact l.c.s. manifold with Lee 1-form $\omega \neq 0$ at every point. Suppose that (Λ, E) is the associated Jacobi structure on M and that there exists a Riemannian metric g on M such that E is Killing with respect to g. Then the cohomology groups $H^k_{\omega B}(M)$ have finite dimension, for $0 \leqslant k \leqslant 2m$.

Proof. Note that $E \neq 0$ at every point. Thus, by renormalizing the metric g, if necessary, we can assume that E is a unit vector field. Consequently, if θ is the 1-form on M given by $\theta(X) = g(X, E)$ for $X \in \mathfrak{X}(M)$, we have that

$$\theta(E) = 1 \qquad \text{and} \qquad \mathcal{L}_E\theta = 0. \tag{60}$$

Let $\mathfrak{I}(M)$ be the isometry group of (M, g). Since M is compact, $\mathfrak{I}(M)$ is a compact Lie group. Moreover, if $\phi : \mathbb{R} \times M \longrightarrow M$ is the flow of the vector field E, $\{\phi_t\}_{t\in\mathbb{R}}$ is a subgroup of $\mathfrak{I}(M)$ whose closure G in $\mathfrak{I}(M)$ is compact, connected and Abelian, hence a torus.

Denote by $T : M \times G \longrightarrow M$ the action of G on M and by $\Omega^k(M)^G$ the subspace of $\Omega^k(M)$ of G-invariant k-forms, that is, $\Omega^k(M)^G = \{\alpha \in \Omega^k(M) | T_a^*\alpha = \alpha, \forall a \in G\}$. We remark that a k-form α on M is G-invariant if and only if $\mathcal{L}_E\alpha = 0$.

Using (54), we deduce that if $\alpha \in \Omega^k(M)^G$ then $\mathrm{d}_\omega\alpha \in \Omega^{k+1}(M)^G$. Therefore, we can introduce the invariant differential complex

$$\cdots \longrightarrow \Omega^{k-1}(M)^G \xrightarrow{\mathrm{d}_\omega} \Omega^k(M)^G \xrightarrow{\mathrm{d}_\omega} \Omega^{k+1}(M)^G \longrightarrow \cdots.$$

We are going to prove that the invariant cohomology groups $H^*_\omega(M)^G$ have finite dimension.

To do this, we consider, for every $0 \leqslant k \leqslant 2m$, the homomorphism $(i_k)^* : H^k_\omega(M)^G \longrightarrow H^k_\omega(M)$ induced by the canonical inclusion $i_k : \Omega^k(M)^G \longrightarrow \Omega^k(M)$. We will show that $(i_k)^*$ is injective, which would imply that $H^k_\omega(M)^G$ has finite dimension (see proposition 10.1).

We orientate G and let ν be the unique bi-invariant volume form on G such that $\int_G \nu = 1$.

Regard $(M \times G, \pi_M, M, G)$ as a trivial, oriented bundle, and let $\pi_G : M \times G \longrightarrow G$ denote the canonical projection. Then, a linear map $I_\nu : \Omega^k(M \times G) \longrightarrow \Omega^k(M)$ is defined by

$$I_\nu\alpha = \int\!\!\!\!\!\!-_G \alpha \wedge \pi_G^*\nu$$

for $\alpha \in \Omega^k(M \times G)$, where $\displaystyle\fint_G$ is the integral over the fibre (see [11, vol I]). Thus, we can define a linear map $p_k = I_\nu \circ T^* : \Omega^k(M) \longrightarrow \Omega^k(M)$ by

$$p_k(\alpha) = \fint_G T^*\alpha \wedge \pi_G^*\nu. \tag{61}$$

We have that $p_k(\alpha) \in \Omega^k(M)^G$ and $p_k \circ i_k = I\,\mathrm{d}_{|\Omega^k(M)^G}$ (see [11, vol II, p 150]).

From (9), we obtain that

$$T^*\omega \wedge \pi_G^*\nu = \pi_M^*\omega \wedge \pi_G^*\nu. \tag{62}$$

On the other hand, if $\alpha \in \Omega^k(M)$ then, using (61), (62) and the properties of the integral over the fibre (see [11, vol I, pp 303, 304]), we deduce that

$$\begin{aligned} p_{k+1}(d_\omega\alpha) &= \fint_G (\mathrm{d}(T^*\alpha \wedge \pi_G^*\nu) + \pi_M^*\omega \wedge T^*(\alpha) \wedge \pi_G^*\nu) \\ &= \mathrm{d}\left(\fint_G T^*\alpha \wedge \pi_G^*\nu\right) + \omega \wedge \fint_G (T^*\alpha \wedge \pi_G^*\nu) = \mathrm{d}_\omega(p_k(\alpha)). \end{aligned}$$

Consequently, the linear map $p_k : \Omega^k(M) \longrightarrow \Omega^k(M)^G$ induces a homomorphism $(p_k)^* : H_\omega^k(M) \longrightarrow H_\omega^k(M)^G$, and a direct computation shows that $(p_k)^* \circ (i_k)^* = I\,\mathrm{d}_{|H_\omega^k(M)^G}$. This implies that $(i_k)^* : H_\omega^k(M)^G \longrightarrow H_\omega^k(M)$ is a monomorphism, and hence $H_\omega^k(M)^G$ has finite dimension.

Finally, we will prove that the space $H_{\omega B}^k(M)$ has finite dimension, for all integers k.

From (60), we obtain that the following sequence of real vector spaces

$$0 \longrightarrow \Omega_B^k(M) \xrightarrow{j_k} \Omega^k(M)^G \xrightarrow{\tau_k} \Omega_B^{k-1}(M) \longrightarrow 0$$

is exact, where j_k is the canonical inclusion and τ_k is the homomorphism defined by $\tau_k(\alpha) = i_E\alpha$, for $\alpha \in \Omega^k(M)^G$. Moreover, a direct computation shows that $\mathrm{d}_\omega \circ j_k = j_{k+1} \circ \mathrm{d}_{\omega B}$ and $-\mathrm{d}_{\omega B} \circ \tau_k = \tau_{k+1} \circ \mathrm{d}_\omega$, for all k. Therefore, we obtain an exact sequence of complexes

$$0 \longrightarrow (\Omega_B^*(M), \mathrm{d}_{\omega B}) \xrightarrow{j} (\Omega^*(M)^G, \mathrm{d}_\omega) \xrightarrow{\tau} (\Omega_B^{*-1}(M), -\mathrm{d}_{\omega B}) \longrightarrow 0$$

which induces a long exact cohomology sequence

$$\cdots \longrightarrow H_{\omega B}^k(M) \xrightarrow{(j_k)^*} H_\omega^k(M)^G \xrightarrow{(\tau_k)^*} H_{\omega B}^{k-1}(M) \xrightarrow{\Delta_k^*} H_{\omega B}^{k+1}(M) \longrightarrow \cdots$$

with connecting homomorphism Δ^*. Thus, using that the space $H_\omega^k(M)^G$ has finite dimension for all k, we deduce that the space $H_{\omega B}^k(M)$ has also finite dimension for all k. □

Remark 10.5. Let $(\bar{M}, \bar{\Phi}, \bar{\eta})$ be a compact cosymplectic manifold such that $[\bar{\Phi}] \in H^2(\bar{M}, \mathbb{Z})$. Suppose that $\pi : M \longrightarrow \bar{M}$ is the principal circle bundle over $\bar{M}$ corresponding to $[\bar{\Phi}]$. Then, M is a compact regular l.c.s. manifold (see example 10.3). Furthermore, if (Λ, E) is the associated Jacobi structure on M, θ is a connection form in $\pi : M \longrightarrow \bar{M}$ and $\bar{g}$ is a Riemannian metric on $\bar{M}$, we have that E is Killing with respect to the Riemannian metric g on M given by $g = \pi^*\bar{g} + \theta \otimes \theta$ (note that E is the infinitesimal generator of the action of S^1 on M). Consequently, from theorem 10.4, we obtain that the cohomology groups $H_{\omega B}^k(M)$ have finite dimension.

Next, we will see that the LJ-cohomology of a particular class of l.c.s. manifolds is completely determined by the basic de Rham cohomology and the cohomology of the subcomplex $(\Omega_B^*(M), \mathrm{d}_{\omega B})$. For this purpose, we will recall some definitions and results.

Let (M, Ω) be a l.c.s. manifold. Denote by $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$. The isomorphism $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ can be extended to a mapping, which we also denote by $\flat$, from the space $\mathcal{V}^k(M)$ onto the space $\Omega^k(M)$ by putting:

$$\flat(X_1 \wedge \ldots \wedge X_k) = \flat(X_1) \wedge \ldots \wedge \flat(X_k) \tag{63}$$

for $X_1, \ldots, X_k \in \mathfrak{X}(M)$. Thus, $\flat$ is also an isomorphism of $C^\infty(M, \mathbb{R})$-modules. Furthermore, if $\# : \Omega^k(M) \longrightarrow \mathcal{V}^k(M)$ is the mapping defined in section 5 (see (18)) then, using (19), (63) and remark 2.1, we obtain

$$\#\alpha = (-1)^k \flat^{-1}(\alpha) \tag{64}$$

for $\alpha \in \Omega^k(M)$.

On the other hand, a vector field X on M is said to be an *infinitesimal automorphism* of (M, Ω) if $\mathcal{L}_X\Omega = 0$. We denote by $\mathfrak{X}_\Omega(M)$ the space of the infinitesimal automorphisms of (M, Ω). If $X \in \mathfrak{X}_\Omega(M)$ and ω is the Lee 1-form of M then, from (7), we deduce that $\mathcal{L}_X\omega = \mathrm{d}(\omega(X)) = 0$, which implies that $\omega(X)$ is constant. Moreover, if $X, Y \in \mathfrak{X}_\Omega(M)$ then $[X, Y] \in \mathfrak{X}_\Omega(M)$. Thus, $\mathfrak{X}_\Omega(M)$ is a Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$ of the vector fields on M (see [29]).

Now, consider the homomorphism $l : \mathfrak{X}_\Omega(M) \longrightarrow \mathbb{R}$ defined by $l(X) = \omega(X)$, for $X \in \mathfrak{X}_\Omega(M)$. We call l the *Lee homomorphism* of $\mathfrak{X}_\Omega(M)$ (see [29]). Since ω is closed, l is a Lie algebra homomorphism for the commutative Lie algebra structure of $\mathbb{R}$ and it is clear that the homomorphism l is trivial or an epimorphism. In the latter case the l.c.s. manifold M is said to be *of the first kind* [29]. We remark that a l.c.s. manifold (M, Ω) is of the first kind if and only if there exists $X \in \mathfrak{X}_\Omega(M)$ such that $l(X) \neq 0$. In fact, we have the following theorem which gives the structure of a l.c.s. manifold of the first kind.

Theorem 10.6 ([29]). Let (M, Ω) be a $2m$-dimensional l.c.s. manifold of the first kind with Lee 1-form ω, and denote by (Λ, E) its associated Jacobi structure. Then, there exists $U \in \mathfrak{X}_\Omega(M)$ such that $l(U) = \omega(U) = 1$ and, if θ is the 1-form on M given by $\theta = -\flat(U)$, we have:

$$\Omega = \mathrm{d}\theta - \omega \wedge \theta \qquad \theta(E) = 1 \qquad i_U\,\mathrm{d}\theta = i_E\,\mathrm{d}\theta = 0 \qquad [E, U] = 0.$$

Moreover, $\omega \wedge \theta \wedge (\mathrm{d}\theta)^{m-1}$ is a volume form on M.

If (M, Ω) is a l.c.s. manifold of the first kind and $U \in \mathfrak{X}_\Omega(M)$ is such that $\omega(U) = 1$ then U is said to be a basic infinitesimal automorphism of (M, Ω).

If $\#_B$ is the restriction of # to the basic forms and σ_I is the LJ-cohomology operator, we obtain the following.

Theorem 10.7. Let (M, Ω) be a l.c.s. manifold of the first kind with Lee 1-form ω. Suppose that $F_k : \Omega^k_B(M) \oplus \Omega^{k-1}_B(M) \longrightarrow \mathcal{V}^k_I(M)$ is the homomorphism of $C^\infty_B(M, \mathbb{R})$-modules given by

$$F_k(\alpha, \beta) = \#_B(\alpha) + U \wedge \#_B(\beta)$$

U being a basic infinitesimal automorphism of (M, Ω). Then:

(i) F_k induces an isomorphism of complexes $F : (\Omega^*_B(M), -\mathrm{d}_B) \oplus (\Omega^{*-1}_B(M), \mathrm{d}_{\omega B}) \longrightarrow (\mathcal{V}^*_I(M), \sigma_I)$.

(ii) For every k, $0 \leqslant k \leqslant \dim M$, we have that $H^k_{\mathrm{LJ}}(M) \cong H^k_B(M) \oplus H^{k-1}_{\omega B}(M)$.

Proof. Using (63) and the fact that $\mathcal{L}_U\Omega = 0$, we deduce that $\mathcal{L}_U \circ \flat = \flat \circ \mathcal{L}_U$. Thus, by (8), it follows that $\mathcal{L}_U\Lambda = 0$ which implies that (see (13))

$$\sigma_I(U) = E \wedge U. \tag{65}$$

Therefore, from (53), (65), theorem 10.6 and the results of section 5, we have that F is a homomorphism of complexes.

On the other hand, since $\mathcal{L}_E \circ \flat = \flat \circ \mathcal{L}_E$, we can define the homomorphism of $C^\infty_B(M,\mathbb{R})$-modules $G_k : \mathcal{V}^k_I(M) \longrightarrow \Omega^k_B(M) \oplus \Omega^{k-1}_B(M)$ by

$$G_k(P) = ((-1)^k(\flat(P) - \theta \wedge i_E\flat(P)), (-1)^k i_E\flat(P))$$

for $P \in \mathcal{V}^k_I(M)$, where θ is the 1-form on M given by $\theta = -\flat(U) = -i_U\Omega$ (see theorem 10.6).

Then, using (19), (63), (64), theorem 10.6 and the fact that $\#(\theta) = U$, we deduce that G_k is the inverse homomorphism of F_k. □

Example 10.8. Let $(\bar{M}, \bar{\Phi}, \bar{\eta})$ be a cosymplectic manifold with Reeb vector field $\bar{\xi}$. Suppose that $[\bar{\Phi}] \in H^2(\bar{M},\mathbb{Z})$ and denote by $\pi : M \longrightarrow \bar{M}$ the principal circle bundle over $\bar{M}$ corresponding to $[\bar{\Phi}]$. We consider on M the 2-form Ω given by (56). Then (M,Ω) is a regular l.c.s. manifold of the first kind (see [7] and example 10.3). In fact, from (56) and (57), we obtain that $U = \bar{\xi}^H$ is a basic infinitesimal automorphism of (M,Ω). Moreover, using theorem 10.7 and the results obtained in example 10.3, we have that $H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{dR}}(\bar{M}) \oplus H^{k-1}_{\bar{\eta}}(\bar{M})$.

In the particular case when $\bar{M}$ is the product of a quantizable symplectic manifold $(\bar{N}, \bar{\Omega})$ with $\mathbb{R}$, we deduce (see (59))

$$H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{dR}}(\bar{N}) \oplus H^{k-1}_{\mathrm{dR}}(\bar{N}). \tag{66}$$

Note that, in this case, if N is the total space of the principal circle bundle over $\bar{N}$ corresponding to $[\bar{\Omega}] \in H^2(\bar{N},\mathbb{Z})$, then $M = N \times \mathbb{R}$ and, from (66) and corollary 7.7, it follows that N is a regular contact manifold and $H^k_{\mathrm{LJ}}(M) \cong H^k_{\mathrm{LJ}}(N)$.

Using the results of [1] and theorems 10.4 and 10.7, we obtain sufficient conditions to ensure the finiteness of the LJ-cohomology groups of a particular class of l.c.s. manifolds of the first kind.

Corollary 10.9. Let M be a $2m$-dimensional compact l.c.s. manifold of the first kind. Suppose that (Λ, E) is the associated Jacobi structure on M and that g is a Riemannian metric on M such that E is Killing with respect to g. Then the LJ-cohomology groups $H^k_{\mathrm{LJ}}(M)$ have finite dimension, for $0 \leqslant k \leqslant 2m$.

In the remainder of this section we will study the relation between the canonical homology and the LJ-cohomology of a l.c.s. manifold of the first kind.

Let M be a $2m$-dimensional l.c.s. manifold of the first kind with Lee 1-form ω and let U be a basic infinitesimal automorphism of M. Denote by $\Omega^k_{BU}(M)$ the subspace of $\Omega^k_B(M)$ given by $\Omega^k_{BU}(M) = \{\alpha \in \Omega^k_B(M)/i_U\alpha = 0\}$.

We define the operator $\mathrm{d}_{BU} : \Omega^k_{BU}(M) \longrightarrow \Omega^{k+1}_{BU}(M)$ by (see [7])

$$\mathrm{d}_{BU}\alpha = \mathrm{d}\alpha - \omega \wedge i_U\,\mathrm{d}\alpha. \tag{67}$$

We have that $\mathrm{d}^2_{BU} = 0$ (see [7]) and thus we can consider the corresponding differential complex $(\Omega^*_{BU}(M), \mathrm{d}_{BU})$ whose cohomology is denoted by $H^*_{BU}(M)$.

In [7], using an adequate star operator, the authors proved that

$$H^{\mathrm{can}}_k(M) \cong H^{2m-k-1}_{BU}(M) \oplus H^{2m-k-2}_{BU}(M). \tag{68}$$

Remark 10.10. Examples of compact l.c.s. manifolds M of the first kind exist such that the LJ-cohomology groups of M have finite dimension and, however, the canonical homology groups have infinite dimension. Consider, for example, the following case: let $(\bar{N}, \bar{\Omega})$ be a $(2m-2)$-dimensional compact quantizable symplectic manifold and $(\bar{\Phi}, \bar{\eta})$ the cosymplectic structure given by (48) on the product manifold $\bar{M} = \bar{N} \times S^1$. If $\pi : M \longrightarrow \bar{M}$ is the principal circle bundle over $\bar{M}$ corresponding to $[\bar{\Phi}]$ then the total space M is a compact regular l.c.s. manifold of the first kind and $U = \bar{\xi}^H$ is a basic infinitesimal automorphism of M, $\bar{\xi}$ being the Reeb vector field of $\bar{M}$ (see example 10.8). Since the canonical homology of a regular Jacobi manifold is isomorphic to the canonical homology of the corresponding quotient Poisson manifold (see [7] and section 6) then, using corollary 8.3 and the results obtained in example 8.4, we deduce that

$$H_k^{\mathrm{can}}(M) \cong (H_{\mathrm{dR}}^{2m-k-1}(\bar{N}) \otimes C^\infty(S^1, \mathbb{R})) \oplus (H_{\mathrm{dR}}^{2m-k-2}(\bar{N}) \otimes C^\infty(S^1, \mathbb{R})).$$

In fact, in [7] it was proved that the cohomology groups $H_{BU}^k(M)$ and $H_{\bar{\xi}}^k(\bar{M})$ are isomorphic.

Now, we return to the general case of a l.c.s. manifold M of the first kind. If $\tilde{\#}_B$ denotes the homomorphism of $C_B^\infty(M, \mathbb{R})$-modules defined by (25) and σ_{IE} is the restriction of the LJ-cohomology operator σ_I to $\mathcal{V}_{IE}^*(M)$, we obtain the following.

Theorem 10.11. Let (M, Ω) be a l.c.s. manifold of the first kind and U a basic infinitesimal automorphism of M. Suppose that (Λ, E) is the associated Jacobi structure on M and that $\tilde{F}_k : \Omega_{BU}^{k-1}(M) \oplus \Omega_{BU}^{k-2}(M) \longrightarrow \mathcal{V}_{IE}^k(M)$ is the homomorphism of $C_B^\infty(M, \mathbb{R})$-modules given by

$$\tilde{F}_k(\alpha, \beta) = \tilde{\#}_B(\alpha) + U \wedge \tilde{\#}_B(\beta).$$

(i) $\tilde{F}_k$ induces an isomorphism of complexes $\tilde{F} : (\Omega_{BU}^{*-1}(M), \mathrm{d}_{BU}) \oplus (\Omega_{BU}^{*-2}(M), -\mathrm{d}_{BU}) \longrightarrow (\mathcal{V}_{IE}^*(M), \sigma_{IE})$.

(ii) For every $k, 0 \leqslant k \leqslant \dim M$, we have that $H_{\mathrm{LJ}E}^k(M) \cong H_{BU}^{k-1}(M) \oplus H_{BU}^{k-2}(M)$, where $H_{BU}^l(M) = \{0\}$ if $l < 0$.

Proof. Proceeding as in the proof of theorem 10.7, we can show that $\tilde{F}$ is an isomorphism of complexes. In fact, the inverse homomorphism of $\tilde{F}_k$ is given by

$$\tilde{F}_k^{-1}(P) = ((-1)^{k-1}(i_U \flat(P) - \theta \wedge i_E i_U \flat(P)), (-1)^{k-2} i_E i_U \flat(P))$$

for $P \in \mathcal{V}_{IE}^k(M)$, where θ is the 1-form on M given by $\theta = -i_U \Omega$. □

Finally, from (68) and theorem 10.11, we conclude with the following.

Corollary 10.12. Let M be a $2m$-dimensional l.c.s. manifold of the first kind. Then the kth canonical homology group $H_k^{\mathrm{can}}(M)$ is isomorphic to the $(2m-k)$th E-LJ-cohomology group $H_{\mathrm{LJ}E}^{2m-k}(M)$.

Remark 10.13. There exist examples of compact l.c.s. manifolds M of the first kind such that the LJ-cohomology groups of M have finite dimension and however the E-LJ-cohomology groups have infinite dimension (see remark 10.10 and corollary 10.12).

11. A non-transitive example

Let $\mathbb{R}^4$ be the four-dimensional Euclidean space. Denote by (x^1, x^2, x^3, x^4) the usual coordinates on $\mathbb{R}^4$ and by $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ the C^∞ real-valued function given by

$$f(x^1, x^2, x^3, x^4) = \sum_{i=1}^{4} (x^i)^2. \tag{69}$$

Consider on $\mathbb{R}^4$ the Jacobi structure (Λ, E) defined by

$$\Lambda = \pi f \left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4} \right)$$
$$E = 2\pi \left(x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^4} - x^4 \frac{\partial}{\partial x^3} \right).$$

If D is the characteristic foliation of $(\mathbb{R}^4, \Lambda, E)$ then a direct computation proves that $D_x = T_x\mathbb{R}^4$ if $x \in \mathbb{R}^4 - \{0\}$ and $D_x = \{0\}$ if $x = 0$. Thus, the characteristic foliation has two leaves: $L_1 = \mathbb{R}^4 - \{0\}$ and $L_2 = \{0\}$. The Jacobi structure $(\Lambda_1, E_1) = (\Lambda_{|L_1}, E_{|L_1})$ induced on the leaf L_1 is g.c.s.. In fact, such a structure is defined by the g.c.s. 2-form Ω_1 on L_1 given by

$$\Omega_1 = \frac{1}{\pi f_1} i_1^*(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4) \tag{70}$$

where $i_1 : \mathbb{R}^4 - \{0\} \longrightarrow \mathbb{R}^4$ is the canonical inclusion and $f_1 = f \circ i_1$. The Lee 1-form ω_1 is

$$w_1 = -\mathrm{d}(\ln f_1) = -\frac{2}{f_1} i_1^*\left(\sum_{i=1}^{4} x^i \, \mathrm{d}x^i \right). \tag{71}$$

Now, we consider the vector field U on $\mathbb{R}^4$ defined by

$$U = -\frac{1}{2} \sum_{i=1}^{4} x^i \frac{\partial}{\partial x^i}.$$

If U_1 is the restriction of U to $L_1 = \mathbb{R}^4 - \{0\}$, a direct computation, using (69)–(71), proves that U_1 is a basic infinitesimal automorphism of (L_1, Ω_1). Therefore, L_1 is a g.c.s. manifold of the first kind. Moreover, (L_1, Λ_1, E_1) is a regular Jacobi manifold. In fact, if we identify L_1 with the product manifold $S^3 \times \mathbb{R}$ via the diffeomorphism

$$F : L_1 \longrightarrow S^3 \times \mathbb{R} \qquad x \longrightarrow \left(\frac{x}{\|x\|}, \ln \|x\| \right)$$

then $F_* E_1$ is the infinitesimal generator of the usual action of S^1 on S^3. Consequently, L_1 and the quotient manifold L_1/E_1 can be identified with the product manifolds $S^3 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$, respectively in such a sense that the canonical projection $\tau : L_1 \longrightarrow L_1/E_1$ is just the mapping $\pi \times I\mathrm{d}_{|\mathbb{R}} : S^3 \times \mathbb{R} \longrightarrow S^2 \times \mathbb{R}$, where $\pi : S^3 \longrightarrow S^2$ is the Hopf fibration.

Thus, using proposition 10.2, theorem 10.7 and the above results, we deduce that

$$\begin{aligned} &H^0_{\mathrm{LJ}}(\mathbb{R}^4 - \{0\}) = \langle [1] \rangle \cong \mathbb{R} \qquad H^1_{\mathrm{LJ}}(\mathbb{R}^4 - \{0\}) = \langle [f_1 U_1] \rangle \cong \mathbb{R} \\ &H^2_{\mathrm{LJ}}(\mathbb{R}^4 - \{0\}) = \langle [\Lambda_1] \rangle \cong \mathbb{R} \qquad H^3_{\mathrm{LJ}}(\mathbb{R}^4 - \{0\}) = \langle [f_1 U_1 \wedge \Lambda_1] \rangle \cong \mathbb{R} \\ &H^4_{\mathrm{LJ}}(\mathbb{R}^4 - \{0\}) = \{0\}. \end{aligned} \tag{72}$$

Note that $[E_1] = [0]$ since

$$E_1 = \sigma_{I1}(-\ln f_1) \tag{73}$$

where σ_I is the LJ-cohomology operator on $\mathbb{R}^4$ and $\sigma_{I1} = (\sigma_I)_{|L_1}$.

Next, we will obtain some results on the LJ-cohomology of the Jacobi manifold $(\mathbb{R}^4, \Lambda, E)$. In particular, we will prove that the LJ-cohomology of $\mathbb{R}^4$ is not isomorphic to the LJ-cohomology of the leaf $L_1 = \mathbb{R}^4 - \{0\}$.

From (72), we have that dim $H^i_{\mathrm{LJ}}(\mathbb{R}^4) \geqslant 1$ for $0 \leqslant i \leqslant 3$. In fact, we deduce:

(i) $[fU] \neq 0$ in $H^1_{\mathrm{LJ}}(\mathbb{R}^4)$;

(ii) $[\Lambda] \neq 0$ in $H^2_{\mathrm{LJ}}(\mathbb{R}^4)$;

(iii) $[fU \wedge \Lambda] \neq 0$ in $H^3_{\mathrm{LJ}}(\mathbb{R}^4)$.

On the other hand, if $g \in C^\infty_B(\mathbb{R}^4, \mathbb{R})$ then (see (13))

$$\sigma_I(g) = -\#(\mathrm{d}g) = \pi f \left(\frac{\partial g}{\partial x^2}\frac{\partial}{\partial x^1} - \frac{\partial g}{\partial x^1}\frac{\partial}{\partial x^2} + \frac{\partial g}{\partial x^4}\frac{\partial}{\partial x^3} - \frac{\partial g}{\partial x^3}\frac{\partial}{\partial x^4} \right)$$

and $\sigma_I(g) = 0$ implies $g_{|L_1} =$ constant. Thus, by continuity, $g =$ constant on $\mathbb{R}^4$, and $H^0_{\mathrm{LJ}}(\mathbb{R}^4) \cong \mathbb{R}$.

Now, we will show that $\dim H^1_{\mathrm{LJ}}(\mathbb{R}^4) \geqslant 2$.

If we suppose that $[E] = [0]$ in $H^1_{\mathrm{LJ}}(\mathbb{R}^4)$ then there exists $g \in C^\infty_B(\mathbb{R}^4, \mathbb{R})$ such that $E = \sigma_I(g)$. By restricting ourselves to $L_1 = \mathbb{R}^4 - \{0\}$ and from (72) and (73), it follows that $\ln f_1 = c - g_1$, c being a constant and $g_1 = g \circ i_1$. But this would imply that the function $\ln f_1$ admits a C^∞-differentiable extension to $\mathbb{R}^4$ which it is not possible. Therefore $[E] \neq [0]$ in $H^1_{\mathrm{LJ}}(\mathbb{R}^4)$. Using (72) and (73), we deduce that $[E]$ and $[fU]$ are independent cohomology classes in $H^1_{\mathrm{LJ}}(\mathbb{R}^4)$.

Finally, we will prove that $\dim H^2_{\mathrm{LJ}}(\mathbb{R}^4) \geqslant 2$.

If we suppose that $[fE \wedge U] = [0]$ in $H^2_{\mathrm{LJ}}(\mathbb{R}^4)$ then there exists $X \in \mathfrak{X}(\mathbb{R}^4)$ such that $[E, X] = 0$ and $fE \wedge U = \sigma_I(X)$. By restricting ourselves to L_1 and from (72) and (73), we obtain that

$$X_1 + (f_1 \ln f_1)U_1 = \sigma_{I1}(g_1) + \lambda f_1 U_1 \tag{74}$$

with $g_1 \in C^\infty_B(L_1, \mathbb{R})$, $\lambda \in \mathbb{R}$ and $X_1 = X_{|L_1}$. Since $\omega_1(\sigma_{I1}(g_1)) = \Lambda_1(\omega_1, \mathrm{d}g_1) = -E_1(g_1) = 0$, we have that (see (71))

$$f_1^2 \ln f_1 = 2\left(\sum_{j=1}^4 x_j X^j\right) \circ i_1 + \lambda f_1^2$$

where $X = \sum_{j=1}^4 X^j \frac{\partial}{\partial x^j}$. But this would imply that the function $f_1^2 \ln f_1$ admits a C^∞-differentiable extension to $\mathbb{R}^4$ which it is not possible. Consequently, $[fE \wedge U] \neq [0]$ in $H^2_{\mathrm{LJ}}(\mathbb{R}^4)$. Using (72) and (73), we conclude that $[\Lambda]$ and $[fE \wedge U]$ are independent cohomology classes in $H^2_{\mathrm{LJ}}(\mathbb{R}^4)$.

Acknowledgment

This work was partially supported through grants DGICYT (Spain) (project PB94-0106) and University of La Laguna.

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